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ON LINEAR COMPLEX APPROXIMATIONS AND EXTENSIONS

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Summary. The paper begins with a survey of publications on numerical complex approximations. Then a descent algorithm is described, the mathematical details of which appear elsewhere. This algorithm is applied to the approximation of different complex Chebyshev polynomials for which the graphs of the moduli are presented. The numerical results suggest a division of complex approximation problems into three classes. For each class it seems appropriate to devise a separate technique in order to obtain better results.

Three extensions of the descent method concern linear approximations of functions with values in a unitary space, nonlinear complex approximations, and L_1-approximations. There are numerical examples for each case.

1. Introduction and Historical Remarks. Recently there has been some interest in computing best complex approximations (with respect to the uniform norm $\|\ \|_\infty$). As examples we quote papers by I. Barrodale [1], J. P. Coleman [2] and G. Elliott [5].

Besides the ordinary function approximation on various domains there are special complex approximations of interest: the determination of complex Chebyshev polynomials (C. Geiger, G. Opfer [6], G. Opfer [14]) and the determination of approximate conformal mappings which leads to the (not so well known) complex approximation problem

$$\|z-(a_2z^2+a_3z^3+\cdots+a_nz^n)\|_\infty=\min,\ n\geqq 2. \tag{1.1}$$

An example is given by W. Krabs, G. Opfer [11]. The main emphasis is here will be on numerical aspects of calculating best complex approximations by means of a descent algorithm which will be summarized in the next section.

There are several other attempts to solve complex approximation problems. One of the first papers which contained several numerical examples was given by J. Williams [18]. He tried to establish a Remez-like algorithm which at least produced "good" approximations by means of rational functions. In order to be able to construct best approximations S. Ellacott and J. Williams suggested another method for the polynomial case [3] and the rational case [4]. The underlying idea in both papers was given by C. L. Lawson [12]. He showed that a certafn weighted L_2-approximation is the same as the corresponding (discrete) uniform approximation. In the rational case S. Ellacott and J. Williams use the local Kolmogorov criterion which is (falsely) assumed to be a sufficient criterion for a best approximation. In a subsequent paper by J. Williams [19] the case where the local Kolmogorov criterion is not sufficient is studied. That the rational complex case in contrast to the rational real case still holds surprises was demonstrated by E. B. Saff and R. S. Varga [15] and M. J. D. Powell as quoted by I. Barrodale [1].

An early paper describing a descent algorithm which internally used an optimization procedure was given by S. I. Zuhovickiĭ, R. A. Poljak and M. E. Primak [20] for the discrete, complex case. A fairly general algorithm for minimizing nondifferentiable functions (as, for instance, the uniform norm) was presented by E. S. Levitin [13].

The fact that the Kolmogorov criterion is a one-sided directional derivative was exploited by W. Krabs [10] to construct a "pseudo gradient" method for finding best approximations. Applications to the construction of digital filters by means of a descent algorithm were given by M. Gutknecht [7]. V. Klotz [9] gave sufficient conditions for best complex approximations by rational functions on the unit disk. R. Schultz [17] developed a descent algorithm which, on the one hand, is applicable to a large family of linear and nonlinear problems in Banach spaces and, on the other hand, is adjustable to the specific problem treated. There are numerous numerical examples for many different types of approximation problems.

This overview is not intended to be complete. But it should be pointed out that the literature on numerical complex approximation is sparse and that even the simplest case, namely the numerical computation of best complex approximation by ordinary polynomials, still may cause difficulties.

2. Linear Complex Approximation. We restrict ourselves here to the case of uniform approximation of a given function f by elements of an n-dimensional linear space $V=\operatorname{span}(v_1, v_2, \ldots, v_n)$. All functions mentioned are assumed to be defined and continuous on a compact subset D of $\mathbb{C}$ and have values in $\mathbb{C}$.

The following descent algorithm is constructed by using the Kolmogorov criterion which in our case characterizes a best approximation. For details compare G. Opfer [14].

Let $\hat{v} \in V$ and $\hat{\delta}>0$ be given.
Determine

$$E_{\hat{v}}(\hat{\delta})=\{z \in D: \|\hat{v}-f\|_{\infty}^{2}-|\hat{v}(z)-f(z)|^{2} \leq \hat{\delta}\} \tag{2.1}$$

and compute a direction

$$v=\sum_{i=1}^{n} a_i v_i \in V; \quad a_i \in \mathbb{C},\ i=1, \ldots, n \tag{2.2}$$

by solving the linear constrained maximum problem

$$\mu-\operatorname{Re}\{\overline{(\hat{v}(z)-f(z))}v(z)\} \leq 0, \text{ for all } z \in E_{\hat{v}}(\hat{\delta}), \tag{2.3a}$$

$$-1 \leq \operatorname{Re} a_i,\ \operatorname{Im} a_i \leq 1, \tag{2.3b}$$

$$\mu=\max. \tag{2.3c}$$

Let μ, v be a solution of (2.3a), (2.3b), (2.3c).

Case I: $\mu=0$.

a) $E_{\hat{v}}(0)=E_{\hat{v}}(\hat{\delta}) \Rightarrow \hat{v}$ is a best approximation of f,

b) $E_{\hat{v}}(0)\neq E_{\hat{v}}(\hat{\delta}) \Rightarrow \hat{\hat{\delta}}=\hat{\delta}2;\ /\hat{\hat{v}}=\hat{v}$.

Case II: $\mu>0$.

a) $\mu\leq\hat{\delta} \Rightarrow \hat{\hat{\delta}}=\hat{\delta}/2;\quad \hat{\hat{v}}=\hat{v}-\hat{\lambda}$

b) $\mu>\hat{\delta} \Rightarrow \hat{\hat{\delta}}=\hat{\delta};\quad \hat{\hat{v}}=v-\hat{\lambda}v$.

The stepsize $\hat{\lambda}$ can be computed optimally as described by G. Opfer [14, formulae (2.12a), (2.11)]. There it is also shown that in Case II we have

$$\|f-\hat{\hat{v}}\|_\infty^2<\|f-\hat{v}\|_\infty^2-\hat{\lambda}\mu. \tag{2.4}$$

Thus we either compute a new iterate $\hat{\hat{v}}, \hat{\hat{\delta}}$ or we stop with a best approximation $\hat{v}$ (Case Ia).

This algorithm was applied in particular to the computation of Chebyshev polynomials

$$T_n(z)=z^n+b_{n-1}z^{n-1}+\cdots+b_0,\ \|T_n\|_\infty=\min \tag{2.5}$$

on various regions D. For D we choose here rectangles and circular sectors. Because the maxima of $|T_n(z)|$ are taken on the boundary ∂D we can replace D by ∂D which simplifies the computations. We started with a very coarse discretization of ∂D (about 10 points) and used the results as starting approximation for finer discretizations (ca. 100 points and ca. 500 points). Even the first computation with few points revealed the qualitative behaviour of the error curve (i. e. $|T_n(z)|$ for $z\in\partial D$) very clearly.

In Figures 1 to 4 we present such error curves for some cases. The horizontal axis of these figures represents the arclength s of ∂D in positive

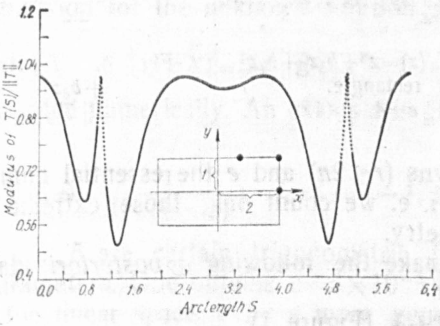

Fig. 1. T-polynomial $T_5(z)=z^5+b_3z^3+b_1z$ on rectangle. $\|T_5\|_\infty=20.37,\ e=3,\ r=2$

orientation starting where ∂D crosses the positive x-axis. The extreme points of $|T_n(z)|$ are marked by dots in the inserted configuration ∂D.

It should be remarked here that in our computations we made one change of the algorithm. Instead of keeping $\hat{\delta}$ fixed (in Case IIb) we enlarged $\hat{\delta}$ suitably. Otherwise it may happen that $E_{\hat{v}}(\hat{\delta})$ remains fixed for all subsequent iterations. Other changes may be useful, too. For details consult R. Schultz [17, p. 19-21]. In general we have n complex unknowns $b_0, b_1, \dots, b_{n-1}$, i. e. $2n$ real unknowns $\operatorname{Re} b_j$, $\operatorname{Im} b_j$, $j=0, 1, \dots, n-1$. But it may happen (for instance, for symmetry reasons) that some or all the unknowns are real, so we have less than $2n$ real unknowns. Call r the num-

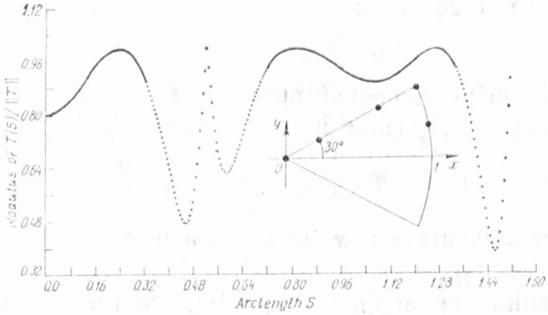

Fig. 2. T-polynomial $T_5(z)=z^5+b_4z^4+b_3z^3+\cdots+b_0$ on circular sector. $\| T_5 \|_\infty=0.02131, e=5, r=5$

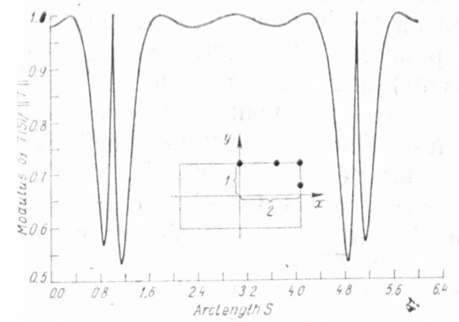

Fig. 3. T-polynomial $T_8(z)=z^8+b_6z^6+b_4z^4+\cdots+b_0$ on rectangle. $\| T_8 \|_\infty=102.1$, $e=4$, $r=4$

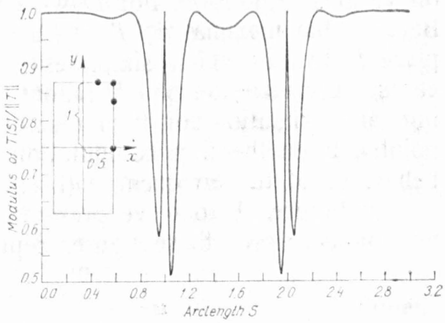

Fig. 4. T-polynomial $T_{10}(z)=z^{10}+b_8z^8+b_6z^6+\cdots+b_0$ on rectangle. $\| T_{10} \|_\infty=0.2964$, $e\geq 4$, $r=5$

ber of real unknowns ($r\leq 2n$) and e the essential number of extreme points of $|T_n(z)|$ on ∂D, i. e. we count only those extreme points which are not produced by symmetry.

Then we can make the following *a posteriori* classification of our numerical results:

Class I: $e\geq r+1$, (Figure 1)

Class II: $e\leq r$, (Figure 2, 3)

Class III: According to the computations we are unable to detect the size of e (Figure 4).

Our computations show, on the one hand, that the qualitative behaviour of the error curve is detected relatively fast but, on the other hand, that the convergence of the coefficients is slow.

In order to improve on the computed results one should set up rules according to the given classification. In Class I, which corresponds to the standard case of real approximation on an interval by elements of a Haar space, the previously mentioned algorithm by J. Williams seems appropriate. In Case II a complex version of an algorithm suggested by R. Hettich [8] should be developed. How to treat cases in Class III remains open.

3. Extensions. There are different types of extensions:

1. The functions $f, v_1, v_2, \ldots, v_n$ introduced in Section 2 are allowed to have values in some unitary space $(H, \langle , \rangle)$ in which the norm may be denoted by $\| \, \|_H$. No change in the stated algorithm has to be made apart from replacing the absolute value by $\| \, \|_H$ and the product $\{(\widehat{v}(z)-f(z))v(z)\}$ in (2.3a) by the scalar product $\langle \widehat{v}(z)-f(z), v(z) \rangle$.

An example would be the approximation of a given kernel $k(t, \tau)$ in a Fredholm integral equation

$$x(t)-\int_I k(t, \tau)\, x(\tau)\, d\tau = g(t), \quad t \in I \tag{3.1}$$

by a degenerate kernel

$$\widetilde{k}(t, \tau) = \sum_{j=1}^{m} r_j(t)\, s_j(\tau)\, ; \quad t, \tau \in I, \tag{3.2}$$

where the functions $r_j(t)$, $s_j(\tau)$ depend on some parameters. I is a given interval and the parameters in r_j, s_j are determined according to

$$\max_{t \in I} \{\int_I (k(t, \tau)-\widetilde{k}(t, \tau))^2 d\tau\}^{1/2} = \min. \tag{3.3}$$

The integral equation

$$\widetilde{x}(t)-\int_I \widetilde{k}(t, \tau)\, \widetilde{x}(\tau)\, d\tau = g(t) \tag{3.4}$$

can be solved explicitly and from the approximation $\widetilde{k}$ by (3.3) one can deduce an error estimation for the unknown solution x of (3.1) in the form

$$\| x-\widetilde{x} \|_\infty \leq c \| g \|_\infty, \tag{3.5}$$

where c can be computed numerically. An example is given by R. Schultz [16, p. 52-54] for

$$k(t, \tau) = \frac{1}{\Pi (1+(\tau-t)^2)}\, ; \quad t, \tau \in I=[-1, 1], \tag{3.6}$$

where $r_j, s_j, j=1, 2, \ldots, 5$ are certain trigonometric functions which contain altogether 7 parameters. One obtains $c=5\times 10^{-3}$ in (3.5).

2. We replace the linear space V by a more general set. In the nonlinear case we make use of the Kolmogorov criterion in its local form and apply the given algorithm locally. At the end of any computation it has to be decided separately whether the computed local solution is locally best or even globally best.

The following numerical examples for complex rational approximations were prepared by R. Schultz [17]. These examples are also treated by S. Ellacott and J. Williams [4].

Example 1. $\| e^z-(a_1+a_2 z+a_3 z^2)/(1+a_4 z+a_5 z^2) \|_\infty = \min$; $a_1, a_2, \ldots, a_5 \in \mathbb{R}$, $D=\{z : |z| \leq 1\}$. The computed solution rounded to four places is:

$$a_1=1.0000,\ a_2=0.5162,\ a_3=0.0921,\ a_4=-0.4838,\ a_5=0.0757,$$

which yields $\|\cdot\|_\infty \leq 1.4\times 10^{-3}$.

Example 2. $\| e^z-(b_1+b_2 z)/(1+b_3 z)\|_\infty=\min$; $b_1, b_2, b_3 \in \mathbb{C}$; $D=\{|z|\leq 1$ $0\leq \arg z \leq \pi/2\}$.

$$b_1=1.0177+i.0.0215,\ b_2=0.6245+i.0.1565,$$

$$b_3=-0.3929+i.0.0552,\ \|\cdot\|_\infty \leq 2.8.10^{-2}.$$

The extreme points of the modulus of the error curve are marked in Fig. 5.

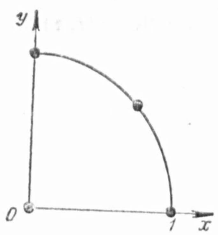

Fig. 5. Extreme points of $|e^z-(b_1+b_2z)/(1+b_3z)|$ on quarter disc D

3. L_1-approximations, for instance, do not fit into our scheme. But it is possible to apply a similar descent algorithm by using a corresponding necessary condition instead of the Kolmogorov criterion. R. Schultz [17, p. 66-68] treated a Fredholm integral equation with the kernel given in (3.6) by approximating it by the degenerate kernel

$$\widetilde{k}_m(t,\tau)=a_1+\sum_{j=1}^{m} a_{2j}\cos(a_{2j+1}(t-\tau))$$

using the operator norm

$$\max_{t\in I}\int_I | k(t,\tau)-\widetilde{k}_m(t,\tau)|\, d\tau=\min.$$

For $m=3$ the constant c from (3.5) is $c=5.1\times 10^{-4}$, in comparison with $c=5\times 10^{-3}$ by the method (3.3).

A locally faster algorithm was developed by K. Glashoff and R. Schültz [21].

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