КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'77
София . 1980, с. 451—453

CONSTRUCTIVE FUNCTION THEORY'77
Sofia . 1980, p. 451—453

SHELTERED POINTS IN NORMED SPACES

P. L. Papini

Summary. The notion of sheltered point for a bounded set in a normed space was introduced by the author to study problems of best approximation by means of hypercircles. Some of the main properties of those points are indicated here.

Let $(X, \|\cdot\|)$ be a real normed space and V a closed subspace of X. If A is a bounded subset of X, we set

$$r_V(A)=\inf\{\sup\{\|x-y\|: x\in A\}: y\in V\};\; r(A)=r_X(A).$$

A *centre* of A [with respect to V] is a point $y\in X$ [$y\in V$] such that $\sup\{\|x-y\|: x\in A\}=r(A)$ [respectively: $\sup\{\|x-y\|: x\in A\}=r_V(A)$]. If $y\in X\setminus V$ and distance $(y, V)=d$, the set $C_s=V\cap B(y, s)$, where $s>d$ and $B(y, s)=\{x\in X;\ \|x-y\|\leq s\}$ is called a *hypercircle*; we set also $P_V(y)=\{x\in V; \|x-y\|=d\}$; $r_V(C_s)=r_s$; $r'_V(C_s)=r'_s$.

If X is a Hilbert space, given any $y\in X\setminus V P_V(y)$ is the center of an arbitrary hypercircle (corresponding to y); this is also a characteristic property of Hilbert spaces (see [3]). The relationship between centers of hypercircles with respect to V and best approximation from V in normed spaces was studied in [1], where a new map was defined (such a map of "approximation" is better behaved than another similar map, introduced in [2] and too particular to be useful). In general the centers of hypercircles may have these faults: non-existence, non-uniqueness, dependence on s, being outside of C_s.

In an attempt to overwhelm these difficulties and to use hypercircles more profitably, the following definitions were given in [4]:

$$r'_V(A)=\sup\{\inf\{\|x-y\|: x\in V\setminus A\}: x\in A\};\; r'(A)=r'_X(A).$$

A *sheltered point* of A [with respect to V] is a point $y\in A$ such that

$$\inf\{\|x-y\|: x\in X\setminus A\}=r'(A) \text{ [respectively: } \inf\{\|x-y\|: x\in V\setminus A\}=r'_V(A)].$$

Set $F(A)=\{x\in A;\ x$ is a sheltered point of $A\}$ (*shelter* of A).

In order to justify the above definitions from the point of view of applications, we give two examples. If we want to give a certain "service" to a set of points where some "clients", are located (e. g. television to a set of towns on a plane region) the best location for the "service station" is the centre of the set (we assume that the station can serve infinitely many clients within a certain distance). But if we want to put something (e. g. a military equipment) in a well protected area, that is as far as possible from "dangerous points" (the boundary of the set), it is preferable to find the shelter of the area.

We recall the main facts proved in [4] for the shelter of a set: of course the sets considered will have a nonempty interior.

Proposition 1 (see [4]). *$F(A)$ is always closed, and it is convex if A is convex; $F(A)\neq\emptyset$ if X is finite dimensional, or if X is reflexive and A is convex; if A is a compact rotund body (so X must be finite dimensional), $F(A)$ is a singleton; $\delta(A)=2r'(A)$ iff $r(A)=r'(A)$, iff $B\subset A\subset\bar{B}$ for an open ball B ($\delta(A)$ denotes the diameter of the set A).*

As well as for centres, the geometry of X shows some relation with the properties of shelters. In general, shelters are not stable, also in nice spaces, as shown by the following

Example. In R^2 the sequence $\{A_n\}$, where $A_n=B(0,1)\setminus\bigcap_{k=n}^{\infty}\{(x,y);\ |x-2^{-k}|<2^{-k+2}\}$ converges (in the sense of Hausdorff) to $A=\bar{B}(0,1)$ (the unit ball); but $r'(A_n)<1/2$, $r'(A)=1$, and the sheltered points of A_n do not converge to $(0,0)$ (which is the unique sheltered point of A).

If we consider convex sets we get also some stability (see [4]). We remark that by using the same definition of shelter for unbounded sets, we should lose nice properties: for example, for the convex set $A=\{(x,y);\ xy\geqq 1,\ 0\leqq y\geqq 2\}\subset R^2$ we should have $r'(A)=1$, $F(A)=\emptyset$.

Proposition 2. *For every bounded set A, $r(A)+r'(A)\leqq\delta(A)$.*

Proof. Given $\varepsilon>0$, let $\bar{x}\in A$ satisfy the inequality $\inf\{\|x-\bar{x}\|:\ x\in X\setminus A\}>r'(A)-\varepsilon$. Since $r(A)\leqq\sup\{\|\bar{x}-y\|:\ y\in A\}$, we can find a point $y'\in A$ such that $\|\bar{x}-y'\|>r(A)-\varepsilon$; now take on the straight line passing through $\bar{x}$ and y' (from the side of $\bar{x}$) a point x' such that $\|x'-\bar{x}\|=r'(A)-\varepsilon$: then $x'\in A$ and

$$\|x'-y'\|=\|x'-\bar{x}\|+\|\bar{x}-y'\|>r'(A)-\varepsilon+r(A)-\varepsilon,$$

so (ε being arbitrary) the thesis follows.

Corollary. *For every hypercircle we have $r_s+r'_s\leqq 2s$. Then, $r_s=s+d$ implies $r'_s=s-d$, and $r'_s=s$ implies $r_s=s$.*

Proof. Consider the sets C_s: the first assertion follows from $\delta(C_s)\leqq 2s$. Moreover (for $s>d$) if $r_s=s+d$ we obtain $s-d\leqq r'_s\leqq 2s-r_s=s-d$; if $r'_s=s$ we have $s=r'_s\leqq r_s\leqq 2s-r'_s=s$.

Now consider $l=\sup\{(r_s-s):\ s>d\}$ and $l'=\sup\{(s-r'_s):\ s>d\}$. It is known that $|l|\leqq d$ and that $0\leqq l'\leqq d$; moreover, $l'=d$ or $l=0$ for every subspace characterize "nice" spaces, while $l'=0$ or $|l|=d$ are pointers of "bad" spaces (see [1] and [4] for these results). These two parameters are also related in the way stated by the following

Proposition 3. *For every pair (y, V) we have $|l|\leqq l'$.*

Proof. By definition, we have for every $s>d$: $-l\leqq s-r_s\leqq s-r'_s\leqq l'$. Moreover, from the above proved Corollary we have: $l=\sup\{(r_s-s):\ s>d\}\leqq\sup\{(s-r'_s):\ s>d\}=l'$.

We conclude recalling that in reflexive spaces, by the shelters of C_s we may construct a new map P'_V, associating to y these points $\bar{y}$ such that $\|\bar{y}-v\| \geqq \|v-y\|-l'$ for every $v \in V \setminus P_V(y)$. P'_V is a "refinement" of P_V: for example, P'_V is single-valued when $l'=0$. If we consider R^2 endowed with the sup norm, the projection of $y=(0,1)$ onto $V=\{(y,0);\ y \in R\}$ is too general, while the point (0,0) (which is the only one in $P'_V(y)$) appears to be a natural selection from $P_V(y)$.

REFERENCES

1. C. Franchetti. Chebyshev centers and hypercircles. *Boll. Un. Mat. Ital.*, (4) 11, 1976, suppl. fasc. 3, 565—573.
2. C. Franchetti, M. Furi. Some characteristic properties of real Hilbert spaces. *Rev. Roumaine Math. Pures Appl.*, 17, 1972, 1045—1048.
3. N. I. Gurarii, Ju. I. Sozonov. Normed spaces in which the unit sphere has no bias. *Math. Notes*, 7, 1970, 187—189.
4. P. L. Papini. Sheltered points in normed spaces. *Ann. Mat. Pura Appl.*, (4), 117, 1978, 233—242.

University of Bologna, Italy.

Received July 29, 1977