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ON AN EMBEDDING THEOREM

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Summary. The problem to be considered is to determine what conditions are necessary and sufficient for space $H_P^\omega[-\pi, \pi]^m$, with modulus of smoothness of an arbitrary order ω, and mixed norm with $P=(p_1, p_2, \ldots, p_m)$, to consist the continuous functions only. The condition is $\sum_{n=1}^{\infty} n^{\sum_{i=1}^{m} 1/p_i - 1}\omega(1/n)<\infty$.

Let m denote a natural number. It is well known that the set $L_P=L_P([-\pi, \pi]^m)$ of all such 2π-periodic (with respect to each variable separately) functions $f\colon R^m \to R$ that $\|f\|_P=\|\ldots\|\|f\|_{p_1}\|_{p_2}\ldots\|_{p_m}<\infty$, where $1\leq P\leq\infty$ (i. e. $1\leq p_i\leq\infty$, $i=1, 2, \ldots, m$), is a Banach space [2]. Any continuous non-decreasing function $\omega\colon [0,2\pi]\to[0, \infty)$, such that $\omega(0)=0$ and $\eta_0^{-s}\omega(\eta_0)\leq c\delta_0^{-s}\omega(\delta_0)$, for a certain natural number s and $c>0$ and every $0<\delta_0<\eta_0<2\pi$, is called the modulus of smoothness of order s.

Let $\omega_P^l\colon [0, 2\pi]^m\times L_P\to R$ be given by $\omega_P^l(\delta, f)=\sup\{\|\Delta_h^l f\|_P\colon |h_i|\leq\delta_i, i=1, 2, \ldots, m\}$, where $h\in R^m$ and the operator Δ_h^l is given by the following recurensive formula: $\Delta_h^0 f=f$, $\Delta_h^l f(x)=\Delta_h^{l-1}f(x+h)-\Delta_h^{l-1}f(x)$, $l=1, 2, \ldots$.

For an arbitrary $K\in R^m$, k_i — natural or zero, $i=1, 2, \ldots, m$, let $\Omega_P^K\colon [0, 2\pi]^m\times L_P\to R$ be given by

$$\Omega_P^K(\delta, f)=\sup\{\|\Xi_h^K f\|_P\colon |h_i|\leq\delta_i, i=1, 2, \ldots m\},$$

where $\Xi_h^K=\Delta_{h_1e^1}^{k_1}\circ\cdots\circ\Delta_{h_me^m}^{k_m}$ and $e^i=(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, 0, \ldots, 0)$ designates a versor in the space R^m, $i=1, 2, \ldots, m$.

Given l — natural and ω — a modulus of smoothness of order s, let $H_P^{\omega, l}=H_P^{\omega, l}([-\pi, \pi]^m)$ denote the set of all such functions f that $f\in L_P$ and $\omega_P^l(\delta, f)=0(\omega(\|\delta\|))$ as $\delta\to 0$, where $\|\cdot\|$ stands for an arbitrary norm in R^m. Let $H_P^\omega=H_P^\omega([-\pi, \pi]^m)=\bigcup_{l=1}^{\infty}H_P^{\omega, l}$; that means that a function $f\in L_P$ belongs to H_P^ω iff there exists l such that $\omega_P^l(\delta, f)=0(\omega(\|\delta\|))$ as $\delta\to 0$. H_P^ω is a linear subspace of L_P.

Our interest is to determine what condition is necessary and sufficient for H_P^ω to consist the continuous functions only. Before answering this question let us introduce a lemma.

We shall use the following abbreviations: if $K=(k_1, k_2, \dots, k_m)$ and $N=(n_1, n_2, \dots, n_m)$, then $K! = k_1!\, k_2! \dots k_m!$, $2^K = 2^{k_1}2^{k_2} \dots 2^{k_m}$, $\alpha/N=(\alpha/n_1, \alpha/n_2, \dots, \alpha/n_m)$ for $\alpha \in R$, $|K| = \Sigma_{i=1}^m |k_i|$, and $K \leq N$ iff $k_i \leq n_i$ for $i=1, 2, \dots, m$. $E_N^P(f)$ will stand for the best approximation of function f in the metric of the space L_P by means of trigonometrical polynomials of order not greater than N.

Lemma 1. *Let* $1 \leq P \leq \infty$ *and* s *be natural. Then for every* $f \in L_P$ *and every* $\delta \in [0, 2\pi]^m$

$$\omega_P^s(\delta, f) \leq \sum_{k_1=0}^{s} \dots \sum_{k_m=0}^{s} \frac{s!}{K!} \Omega_P^K(\delta, f).$$

Proof. First, let us prove by induction for s that

$$(1) \qquad \Delta^s_{\sum_{i=1}^n h_i} f(x) = \sum_{i_1=1}^{n} \dots \sum_{i_s=1}^{n} \Delta_{h_{i_1}} \circ \dots \circ \Delta_{h_{i_s}} f\Big(x + \sum_{\nu=1}^{s} \sum_{j_\nu=1}^{i_\nu - 1} h_{j_\nu}\Big).$$

Indeed, if $s=1$, it is simple to verify, and supposing for the moment that $\Sigma h_i = \sum_{i=1}^n h_i$, we have

$$\Delta^{s+1}_{\Sigma h_i} f(x) = \Delta^s_{\Sigma h_i}(\Delta_{\Sigma h_i} f(x))$$

$$= \sum_{i_1=1}^{n} \dots \sum_{i_s=1}^{n} \Delta_{h_{i_1}} \circ \dots \circ \Delta_{h_{i_s}} \Big(\sum_{i_{s+1}=1}^{n} \Delta_{h_{i_{s+1}}} f\Big(x + \sum_{\nu=1}^{s} \sum_{j_\nu=1}^{i_\nu-1} h_{j_\nu} + \sum_{j_{s+1}=1}^{i_{s+1}-1} h_{j_{s+1}}\Big)\Big),$$

which gives us (1).

Taking into account that the composition of the operators Δ_{h_i} satisfies either commutative or associative laws, we have

$$\|\Delta^s_{\Sigma h_i} f\|_P \leq \sum_{i_1=1}^{n} \dots \sum_{i_s=1}^{n} \|\Delta_{h_{i_1}} \circ \dots \circ \Delta_{h_{i_s}} f\|_P$$

$$= \sum_{\substack{k_1=0 \\ |K|=s}}^{s} \dots \sum_{k_m=0}^{s} \frac{s!}{K!} \|\Delta^{k_1}_{n_1} \circ \dots \circ \Delta^{k_m}_{h_m} f\|_P.$$

Applying the above inequality to $h = \sum_{i=1}^m h_i e^i$, we have

$$\|\Delta_h^s f\|_P \leq \sum_{\substack{k_1=0 \\ |K|=s}}^{s} \dots \sum_{k_m=0}^{s} \frac{s!}{K!} \|\Xi_h^k f\|_P,$$

which allows us to complete the proof.

Remark 1. The lemma cannot be improved in the sense that there exists a function $f \in L_P$ for which the inequality changes into equality (e. g. $f(x) = f_1(x_1)$, $f_1 \in L_{P1}[-\pi, \pi]$).

Corollary 1. *Under the assumptions as in the lemma there exists a constant c that for $l \geq mr$*

$$\omega_P^l(\delta, f) \leq c \sum_{i=1}^{m} \omega_{i, P}^r(\delta_i, f),$$

where $\omega_{i, P}^r(\delta_i, f) = \omega_P^r(\delta_i e^i, f) = \Omega_P^{re^i}(\delta, f)$.

Proof. Let us first notice that for every $\delta \in [0, 2\pi]^m$, $f \in L_P$ and K, L, k_i, l_i — natural or zero, we have

$$\Omega_P^{K+L}(\delta, f) \leq 2^K \Omega_P^L(\delta, f). \tag{2}$$

By the lemma we get

$$\omega_P^l(\delta, f) \leq \underset{|K|=l}{\sum_{k_1=0}^{l} \ldots \sum_{k_m=0}^{l}} \frac{l!}{K!} \Omega_P^K(\delta, f).$$

As $|K| = l$ and $l \geq mr$, so for at least one j we have $k_j \geq r$, and by (2) we can estimate

$$\Omega_P^K(\delta, f) \leq 2^{|K|-r} \Omega_P^{re^j}(\delta, f) = C_K \omega_{j, P}^r(\delta_j, f),$$

which completes the proof.

Corollary 2. *Let ω denote a modulus of smoothness of order s, $1 \leq P \leq \infty$. Then $f \in H_P^\omega$ iff there exists a natural number l that $\omega_{i, P}^l(\delta_0, f) = O(\omega(\delta_0))$ for $i = 1, 2, \ldots, m$.*

Proof. ($\Rightarrow$) If $f \in H_P^\omega$, then $\bigvee_l f \in H_P^{\omega, l}$, and since $\omega_{i, P}^l(\delta_0, f) \leq \omega_P^l(\delta, f)$ $\delta = (\delta_0, \delta_0, \ldots, \delta_0)$, we have

$$\omega_{i, P}^l(\delta_0, f) = O(\omega(\|\delta\|)) = O(\omega(\delta_0)),$$

for $\omega(c\delta_0) \leq (c+1)^s \omega(\delta_0)$ (ref. [6, p. 116]).

($\Leftarrow$) This implication is due to corollary 1.

Remark 2. The space H_P^ω introduced here is a generalization of that of *N.* Temigraliev [5].

Now, we shall introduce the main result of this paper.

Theorem. *Let ω denote a modulus of smoothness of order s, $1 \leq P < \infty$, and $1 \leq \sum_{i=1}^m p_i^{-1} < s$. Then $H_P^\omega \subset C$ iff*

$$\sum_{n=1}^{\infty} n^{\sum_{i=1}^{m} p_i^{-1} - 1} \omega(1/n) < \infty. \tag{3}$$

Proof. ($\Leftarrow$) This implication can be proved almost in the same way as in [1, p. 146] when taking into account modulus of smoothness instead of modulus of continuity. Indeed, for every function $f \in H_P^\omega$ by lemma 1 from [3, p. 86] we obtain $E_N^P(f) \leq c\omega_P^r(1/N, f)$, $1/N = (1/n, 1/n, \ldots, 1/n)$, $r = 1, 2, \ldots$ and accordingly to (3) we have

$$n^{\sum_{i=1}^{m} p_i^{-1}} \omega_P^r(1/N, f) \to 0, \quad \text{as } n \to a,$$

and

$$\sum_{l=m+1}^{\infty} l^{\sum_{i=1}^{m} p_i^{-1}-1} \omega_P^r(1/L, f)\to 0, \quad \text{as} \quad n\to\infty,$$

where $L=(l, l, \ldots, l)$.
Consequently

$$E_N^{(\infty,\ldots,\infty)}(f)\leq C_1 [n^{\sum_{i=1}^{m} p_i^{-1}} E_N^P(f)+ \sum_{l=n+1}^{\infty} l^{\sum_{i=1}^{m} p_i^{-1}-1} E_L^P(f)]\to 0$$

as $n\to\infty$, that means that f is equivalent to a continuous function. This gives us the proof of sufficiency.

($\Longrightarrow$) On the contrary, let us assume that condition (3) holds and there exists P, $1\leq P<\infty$, $1\leq \sum_{i=1}^{m} p_i^{-1}<s$, such that

$$\sum_{n=1}^{\infty} n^{\sum_{i=1}^{m} p_i^{-1}-1} \omega(1/n)=\infty. \tag{4}$$

We shall point a function $F\in H_P^{\omega}$ to be essentially unbounded.

On the basis of lemma 5 from [5, p. 105], as (4) holds, there exists a sequence $(B_n)_{n=1}^{\infty}$, $B_n\downarrow 0$ as $n\to\infty$, such that

$$\sum_{n=1}^{\infty} n^{\sum_{i=1}^{m} p_i^{-1}-1} B_n=\infty,\ B_n\leq\omega(1/n),\ \sum_{k=1}^{n} k^{s-1}B_k\leq cn^s B_n, \tag{5}$$

and

$$\sum_{n=1}^{\infty} 2^{n \sum_{i=1}^{m} p_i^{-1}} (B_{2^n}-B_{2^{n+1}})=\infty. \tag{6}$$

Let us consider a sequence of functions $g_n\colon [-\pi, \pi]\to R$ of one variable x_i given by

$$g_n(x_i)=\begin{cases} 4\dfrac{2^{n-2}+1}{2^{n-3}+1}[\mathfrak{K}_{2^{n-2}}(x_i)\cos(3\,.\,2^{n-2}x_i)-\dfrac{1}{2(2^{n-2}-1)}\cos(2^n x_i)]\text{for } p_i=1,\\ 2^{n(1/p_i-1)}\displaystyle\sum_{k=1}^{2^{n-1}}\cos(2^{n-1}+k)x_i,\ \text{for } p>1.\end{cases} \tag{7}$$

In this definition $\mathfrak{K}_n$ stands for Fejer's kernel (ref. [8, p. 148]) with properties:

$$\mathfrak{K}_n(0)=(n+1)/2,\ \frac{1}{\pi}\int_{-\pi}^{\pi}\mathfrak{K}_n(x_i)dx_i=1,\quad \mathfrak{K}_n(x_i)\geq 0.$$

It has been shown in [4] that $\|g_n\|p_i\leq c$ provided $p_i=1$. Supposing that $1<p_i<\infty$, since

$$\|\sum_{k=1}^{2^{n-1}}\cos(2^{n-1}+k)x_i\|p_i\leq c_i 2^{n(1-1/p_i)} \text{ (ref. [7, p. 209])}$$

we have $\| g_n \|_{p_i} \leq c$. Hence

(8) $$\| g_n \|_{p_i} \leq c, \quad 1 \leq p_i < \infty, \quad n = 2, 3, \ldots.$$

As $\| \Sigma_{n=k}^{\infty} (B_{2^n} - B_{2^{n+1}}) g_n \|_{p_i} \leq c B_{2^k}$, then by completeness of the space L_{p_i}, the function

(9) $$F_i(x_i) = \sum_{n=1}^{\infty} (B_{2^n} - B_{2^{n+1}}) g_n(x_i)$$

is well defined and belongs to L_{p_i}. Consequently, the function $F(x) = F_i(x_i)$ is well defined and belongs to L_P.

Now, we intend to prove that $F \in H_P^{\omega}$. Similarly to [1, p. 145], let n denote an arbitrary natural number. There exists ν, natural or zero, that $2^{\nu} < n \leq 2^{\nu+1}$. Directly from definition (7), using (8) and the fact that every function g_n can be rewritten in the form

(10) $$g_n(x_i) = \sum_{k=2^{n-1}+1}^{2^n} \beta_k \cos kx_i,$$

with certain $\beta_k \geq 0$, we obtain the following estimation of $E_n^{p_i}(F_i)$:

$$E_n^{p_i}(F_i) \leq E_{2^{\nu}}^{p_i}(F_i) \leq \| \sum_{n=\nu+1}^{\infty} (B_{2^n} - B_{2^{n+1}}) g_n \|_{p_i} \leq c B_{2^{\nu+1}} \leq c B_n.$$

For this reason and by 6.1(1) from [6, p. 344] we have

$$\omega_P^r(1/N, F) = C_P \, \omega_{p_i}^r(1/n_i, F_i)$$

$$\leq n_i^{-r} C_P C_r \sum_{\nu=0}^{n_i} \nu^{r-1} E_{\nu}^{p_i}(F_i) \leq n_i^{-r} c n_i^r B_{n_i} \leq c B_{n_i} \leq c \omega(1/n_i).$$

Thus $F \in H_P^{\omega}$.

As $F \in L_P$, then the series (9) is the Fourier series of function F. Moreover, $g_n(0) = 2^{n/p_i}$. According to (10), we get the following two properties of sum of order $N = (n, n, \ldots, n)$ of the Fourier series at the origin: $S_N(0, F) \geq 0$ for every N, and $S_N(0, F) \to \infty$ as $n \to \infty$. It has been shown in [1, p. 145] that these are sufficient for F to be essentially unbounded. So, the proof of the theorem is completed.

Remark 3. If $\Sigma_{i=1}^m 1/p_i \geq s$, then (3) implied $\omega \equiv 0$, and therefore, H_P^{ω} consists of the constant functions only.

Remark 4. This problem was first mentioned by Ya. L. Geronimus (1958) who got the sufficient condition for the imbedding. Then P. L. Uljanov [7] showed that condition (3) is not only sufficient but necessary as well. These two results have been done for the case of one variable and the modulus of smoothness of order 1. N. Temigraliev (among others in [5]) generalized the result of Uljanov for the case of m variables. The theorem of this paper as well as the result of [1] are generalizations of all the results that have just been mentioned.

REFERENCES

1. J. Albrycht, F. Perz. On an embedding theorem and uniform convergence of multiple Fourier series. *Functiones et approximatio*, 4, 1976, 143—148.
2. A. Benedek, R. Panzone. The spaces L_P, with mixed norm. *Duke Math. J.*, 28, 1961, Nr 3, 301—324.
3. G. Kozłowska. Aproksymacja funkcji dwu zmiennych funkcjami całkowitymi. *Zeszyty Nauk. Polit Ślaskiej*, 24, 1974, Nr 386, 79—111.
4. С. Б. Стечкин. Об абсолютной сходимости рядов Фурие. *Известия АН СССР, сер. мат.*, 17, 1953, 87—98
5. Н. Темиргралиев. Об одной теореме вложения. *Известия ВУЗ, сер. мат.*, 7, 1973, 103—111.
6. А. Ф. Тиман. Теория приближения функций действительного переменного. Москва, 1960.
7. П. Л. Ульянов. Вложение некоторых классов функций H_r^ω. *Известия АН СССР, сер. мат.*, 32, 1968, 649—686.
8. A. Zygmund. Trigonometric series, vol I. Cambridge, 1959.

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