КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'77
София . 1980, с. 465—468

CONSTRUCTIVE FUNCTION THEORY'77
Sofia . 1980, p. 465—468

ON THE ONE-SIDED APPROXIMATION OF FUNCTIONS

V. A. Popov

Summary. Let the function f be defined on the interval $[a, b]$ and let us set $\omega_k(f, x; \delta) = \sup |\Delta_h^k f(t)|$, $t, t+kh \in [x-k\delta/2, x+k\delta/2] \cap [a, b]$, where

$$\Delta_h^k f(t) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(t+mh).$$

The following new modulus of the function f is considered: $\tau_k(f; \delta)_{L_p} = \| \omega_k(f, x; \delta) \|_{L_p[a,b]}$. There are given direct and converse theorems for one-sided trigonometrical approximation of functions in L_p. An application of the modulus $\tau_k(f; \delta)_{L_1}$ in order to estimate the error of the Newton Cotes quadrature formulas is also given.

The purpose of this paper is to introduce a new characteristic of the functions and to show that this characteristic is useful in problems, connected with one-sided approximation of functions, and also for estimation of the error of the Newton-Cotes composite quadrature formulas.

Let the function f be defined on the interval $[a, b]$. We set:

$$\omega_k(f, x; \delta) = \sup \{ |\Delta_h^k f(t)| : t, t+kh \in [x-k\delta/2, x+k\delta/2] \cap [a, b] \},$$

where

$$\Delta_h^k f(t) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(t+mh).$$

The characteristic — the modulus — of function f, which we shall consider here, is the following:

(1) $$\tau_k(f; \delta)_{L_p[a,b]} = \| \omega_k(f, x; \delta) \|_{L_p[a,b]}.$$

Modulus of the type (1) for $k=1$ was considered, as we know, by B. Sendov [1] and P. P. Korovkin [2] first. The case $k=1$, $p=1$ is considered by E. P. Dolženko and E. A. Sevastianov [3], where many properties of $\tau_1(f; \delta)_{L_1}$ are given. In [4] B. Sendov uses the modulus $\tau_1(f; \delta)_{L_p}$ for the study of the convergence of sequences of linear positive operators.

This modulus has properties which are similar to the properties of the usual modulus of continuity in L_p (for simplification we consider the 2π-periodical case):

$$\omega_k(f;\delta)_{L_p}=\sup\{(\int_0^{2\pi}|\Delta_h^k f(x)|^p dx)^{1/p}: 0<h\leq\delta\},\ 1\leq p\leq\infty.$$

Lemma 1. *Let f and g be 2π-periodical functions and $f, g\in L_p[0, 2\pi]$, $1\leq p\leq\infty$. Then*

1) $\tau_k(f;\delta)_{L_p}\leq\tau_k(f;\delta')_{L_p},\ \delta\leq\delta'$;
2) $\omega_k(f;\delta)_{L_p}\leq\tau_k(f;\delta)_{L_p}$;
3) $\omega_k(f;\delta)_{L_\infty}=\tau_k(f;\delta)_{L_\infty}$;
4) $\tau_k(f+g;\delta)_{L_p}\leq\tau_k(f;\delta)_{L_p}+\tau_k(g;\delta)_{L_p}$;
5) $\tau_k(f;\delta)_{L_p}\leq\delta\tau_{k-1}(f';\delta)_{L_p}$;
6) $\tau_1(f;\delta)_{L_p}\leq\delta\|f'\|_{L_p}$;
7) $\tau_k(f;\lambda\delta)_{L_p}\leq(1+c_1\lambda)^{c_2k}\tau_k(f;\delta)_{L_p}$, *where c_1 and c_2 are constants (we do not know its exact value now).*

Other properties of $\tau_k(f;\delta)_{L_p}$ are given in the paper of A. Andreev in this volume p. 205—208.

Let T_n be the set of all trigonometrical polynomials of n-th order. The best one-sided approximation of the function $f\in L_p(0, 2\pi)$ in L_p by means of trigonometrical polynomials of n-th order is given by $\widetilde{E}_n^T(f)_{L_p}=\inf\|P-Q\|_{L_p}$, $P, Q\in T_n$, $Q(x)\leq f(x)\leq P(x)$ for all x.

There are many works on the one-sided approximation of functions (see for example [5]-[12]). Let us mention only the result of T. Ganelius [6]:

$$\widetilde{E}_n^T(f)_{L_1}\leq c(k)\,n^{-k-1}\quad V_0^{2\pi}f^{(k)},$$

where $V_0^{2\pi}f^{(k)}$ denotes the variation of the k-th derivative $f^{(k)}$ of the function f, and the result of Babenko and Ligun [7]:

$$\sup\{\widetilde{E}_n^T(f)_{L_p}: f\in W^k L_p\}=0\,(n^{-k}),\quad k=1, 2, \ldots,$$

($f\in W^k L_p$ if $f^{(k-1)}$ is absolutely continuous and $\|f^{(k)}\|_{L_p}\leq 1$).

It is easy to see that it is not possible to obtain Jackson's type theorems for the best one-sided approximation by means of the usual modulus of continuity $\omega_k(f;\delta)_{L_p}$. But using the modulus $\tau_k(f;\delta)_{L_p}$ it is possible to obtain direct (Jackson's type) and converse (Bernstein's type) theorems for the best one-sided approximation.

The direct theorem is the following:

Theorem 1. *Let f be a bounded 2π-periodical function. Then for $1\leq p\leq\infty$ we have: $\widetilde{E}_n^T(f)_{L_p}\leq c(k)\,\tau_k(f;n^{-1})_{L_p}$.*

For the proof see [13], [14] (see also the cited paper of A. Andreev).

The converse theorem for the best one-sided trigonometrical approximation we shall give in the following form (Salem-Steckin form of Bernstein's theorem):

Theorem 2. *We have:* $(1\leq p\leq\infty)$

$$\tau_k(f;n^{-1})_{L_p}\leq c_1(k)\,n^{-k}\sum_{\delta=0}^{n}\delta^{k-1}\widetilde{E}_s^T(f)_{L_p}.$$

The constants $c(k)$ and $c_1(k)$ in theorems 1 and 2 depend only on k.

The proof of theorem 2 is given in [15].

We see that it is possible to obtain the full characterization of the best one-sided trigonometrical approximation by means of the modulus $\tau_k(f;\delta)_{L_p}$: from theorem 1 and 2 it follows, as usual.

Corollary. *We have* $\tau_k(f;\delta)_{L_p}=0(\delta^\alpha)$, $k-1\leq\alpha<k$, *if and only if* $\widetilde{E}_n^T(f)_{L_p}=0(n^{-\alpha})$, $1\leq p\leq\infty$.

It is possible to obtain also direct and converse theorems for the best one-sided approximation of functions by means of splines, using the modulus $\tau_k(f;\delta)_{L_p}$. The direct theorem is as theorem 1, but we have not now the final converse theorem for one-sided spline approximation.

Remark. Obviously in the case $p=\infty$ theorems 1 and 2 are the well-known theorems for the best uniform approximation (see lemma 1, property 2).

Now we shall give an application of the modulus $\tau_k(f;\delta)_{L_1}$ in order to estimate the error of the Newton-Cotes quadrature formulas.

Theorem 3. *The following estimations are true:*

a) *for the triangular rule:*

$$\left|\int_0^1 f(x)dx-\frac{1}{n}\sum_{i=1}^{n} f((2i-1)/2n)\right|\leq c_1\tau_2(f;n^{-1})_{L_1}; \tag{2}$$

b) *for trapezoidal rule:*

$$\left|\int_0^1 f(x)dx-(1/2n)\left(f(0)+2\sum_{i=1}^{n-1} f(i/n)+f(1)\right)\right|\leq c_2\tau_2(f;n^{-1})_{L_1}; \tag{3}$$

c) *for Simpson's rule:*

$$\left|\int_0^1 f(x)dx-(1/6n)\left(f(0)+2\sum_{i=1}^{n-1} f(i/n)+4\sum_{i=1}^{n-1} f((2i-1)/2n)+f(1)\right)\right| \tag{4}$$

$$\leq c_3\tau_4(f;n^{-1})_{L_1}.$$

Estimations of this type are true for the general composite quadrature formulas of Newton-Cotes type.

The proof of (3) is immediate:

$$\left|\int_0^1 f(x)\,dx-(1/2n)\left(f(0)+2\sum_{i=1}^{n-1} f(i/n)+f(1)\right)\right|\leq\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left|f(x)-\left\{f\left(\frac{i-1}{n}\right)\right.\right.$$

$$\left.\left.+(nx-i+1)\left(f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right)\right\}\right|dx\leq\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\omega_2(f,x;n^{-1})\,dx=\tau_2(f;n^{-1})_{L_1}.$$

The proofs of (2) and (4) are more complicated.

REFERENCES

1. Бл. Сендов. Аппроксимация относительно хаусдорфова расстояния. Докторская диссертация, Москва, 1968.
2. П. П. Коровкин. Опыт аксиоматического построения некоторых вопросов теории приближения. *Ученые зап. Калинингр. пед. инст.*, 69, 1969, 91—109.

3. Е. П. Долженко, Е. А. Севастьянов. О приближениях функций в хаусдорфовой метрике посредством кусочно монотонных (в частности рациональных) функций. *Мат. сборник*, 101(143), 1976, 508—541.
4. Bl. Sendov. Convergence of sequences of monotonic operators in *A*-distance. *C. R. Acad. bulg. Sci.*, 30, 1977, 657—660.
5. G. Freud. Über einseitige Approximation durch Polynome I. *Acta Sci. Math.* 16, 1955, 12—18.
6. T. Ganelius. On one-sided approximation by trigonometrical polynomials. *Math. Scand.*, 4, 1956, 247—258.
7. В. Ф. Бабенко, А. А. Лигун. Порядок наилучших односторонних приближений полиномами и сплайнами в метрике L_p. *Мат. заметки*, 19, 1976, 323—329.
8. В. Г. Доронин, А. А. Лигун. Наилучшее одностороннее приближение некоторых классов дифференцируемых периодических функций. *Доклады АН СССР*, 230, 1976, № 1, 19—21.
9. В. Г. Доронин, А. А. Лигун. Верхние грани наилучших односторонних приближений сплайнами классов $W^r L_1$. *Мат. заметки*, 19, 1876, № 1, 11—17.
10. Г. Фройд, В. А. Попов. Некоторые вопросы, связанные с аппроксимацией сплайн-функциями и многочленами. *Studia Sci. Math. Huug.*, 5, 1970, 161—171.
11. A. Meir, A. Sharma. One-sided spline approximation. *Studia Sci. Math. Hung.*, 31, 1968, 211—218.
12. G. Freud. On the theory of one-sided weighted L_1-approximation by polynomials. Linear Operators Approximation II. Basel, 1976, 285—303.
13. A. S. Andreev, V. A. Popov, Bl. Sendov. Jackson's type theorems for one-sided polynomial and spline approximation. *C. R. Acad. bulg. Sci.*, 30, 1977, 1533—1536.
14. V. A. Popov, A. S. Andreev. Stečkin's type theorems for one-sided trigonometrica. and spline approximation. *C. R. Acad. bulg. Sci.*, 31, 1978, 151—154.
15. V. A. Popov. Converse theorem for one-sided trigonometrical approximation. *C. R Acad. bulg. Sci.*, 30, 1977, 1529—1532.

Centre for Mathematics
and Mechanics P. O. Box 373
1090 Sofia Bulgaria

Received October 19, 1977