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COMPLETENESS OF LAGUERRE AND HERMITE FUNCTIONS OF SECOND KIND

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Summary. Let $\{M^{(\alpha)}_{n_k}(z)\}_{k=0}^{\infty}$ and $\{G_{n_k}(z)\}_{k=0}^{\infty}$ be subsequences of the systems of Laguerre, resp. Hermite functions of second kind. In the paper some results about the completeness of such subsequences in different spaces of analytic functions are given in terms of the density of the sequence $\{n_k\}_{k=0}^{\infty}$.

The system of the Laguerre functions of second kind $\{M^{(\alpha)}_n(z)\}_{n=0}^{\infty}$ $(\alpha>-1)$ is defined by the equalities $(z\in\mathbb{C}-[0,+\infty))$

$$(1)\qquad M^{(\alpha)}_n(z)=-\int_0^{\infty}(t-z)^{-1}t^{\alpha}\exp(-t)L^{(\alpha)}_n(t)\,dt,\ n=0,1,2,\dots,$$

where $\{L^{(\alpha)}_n(z)\}_{n=0}^{\infty}$ are the Laguerre polynomials with parameter α. Respectively the system of Hermite functions of second kind $\{G_n(z)\}_0^{\infty}$ is defined as follows $(z\in\mathbb{C}-(-\infty,+\infty))$

$$(2)\qquad G_n(z)=-\int_{-\infty}^{\infty}(t-z)^{-1}\exp(-t^2)H_n(t)\,dt,\ (n=0,1,2,\dots),$$

where $\{H_n(z)\}_0^{\infty}$ are the Hermite polynomials.

Using the asymptotic formulas for these two systems [1], [2], it is not difficult to describe the region of convergence of a series of the kind $\Sigma_{n=0}^{\infty}b_nM^{(\alpha)}_n(z)$, resp. $\Sigma_{n=0}^{\infty}b_nG_n(z)$.

For series in the Laguerre functions of second kind region of convergence $\Delta^*(\mu_0)$ is defined by the inequality $\mathrm{Re}(-z)^{1/2}>\mu_0=\max\{0,\overline{\lim}_{n\to+\infty}(2\sqrt{n})^{-1}\ln|b_n|\}$. For series in Hermite functions of second kind, the corresponding "region" of convergence is defined by the inequality $|\mathrm{Im}z|>\tau_0=\max\{0,\overline{\lim}_{n\to+\infty}(2n+1)^{-1/2}\ln|(2n/e)^{nk}b_n|$.

In our paper [3] we proved that the system (1) is complete in the space of the complex functions analytic in the region $\Delta^*(\mu_0)$. A statement of this kind is given also for the system (2) in [2, Theorem 8].

In this paper we consider the problem of completeness of the systems $\{M_{n_k}^{(\alpha)}(z)\}_{k=0}^{\infty}$ and $\{G_{n_k}(z)\}_{k=0}^{\infty}$. We make use of the same methods and ideas as in the paper [4], where an analogous question is discussed for the systems $\{H_{n_k}(z)\}_{k=0}^{\infty}$ and $\{L_{n_k}^{(\alpha)}(z)\}_{k=0}^{\infty}$.

Throughout the paper we use the following notation namely if $G\subset\mathbb{C}$ is a region, with $A(G)$ is denoted the space of all complex functions analytic in G with the usual topology of uniform convergence on every compact subset of G.

Theorem 1. *Let* $\alpha>-1$, $\operatorname{Re} w_0<0$ *and* $D_{w_0}:=\{w\in\mathbb{C}:|w-w_0|<|w_0|\}$. *If* $\sigma=\lim\limits_{k\to+\infty}(k/n_k)>0$, *the system* $\{M_{n_k}^{(\alpha)}(z)\}_{k=0}^{\infty}$ *is complete in* $A(D_{w_0})$.

Proof. Let H_{w_0} be the half-plane defined by the inequality $\operatorname{Re}(-w_0/w)<0$ and l_{w_0} be the ray $\{\zeta\in\mathbb{C}:\zeta=-w_0t,\ 0\leq t<+\infty\}$. If $z\in\mathbb{C}-l_{w_0}$ and $w\in H_{w_0}$, we put

(3)
$$\Phi_\alpha(z,w)=-(w/w_0)^{-1-\alpha}\int\limits_{l_{w_0}}(\zeta-z)^{-1}\zeta^\alpha\exp(-w_0\zeta/w)\,d\zeta$$

and get a complex function which is analytic in the region $(\mathbb{C}-l_{w_0})\ xH_{w_0}$.

Now we are going to prove that if $w\in D_{w_0}$ and $z\in H_{w_0}$

(4)
$$\Phi_\alpha(z,w)=\sum_{n=0}^{\infty}(-1)^n w_0^{-n}M_n^{(\alpha)}(z)(w-w_0)^n.$$

Using Cauchy integral theorem, from (1) one can easily get that if $z\in H_{w_0}$

(5)
$$M_n^{(\alpha)}(z)=-\int\limits_{l_{w_0}}(\zeta-z)^{-1}\zeta^\alpha\exp(-\zeta)\,L_n^{(\alpha)}(\zeta)d\zeta.$$

It is well known that the system of the Laguerre polynomials can be definedbymeans of a generating function, namely, if $|\eta|<1$, $\sum_{n=0}^{\infty}L_n^{(\alpha)}(\zeta)\eta^n=(1-\eta)^{-\alpha-1}\exp[-\zeta\eta(1-\eta)^{-1}]$. Putting $\eta=-(w-w_0)/w_0$ and having (5) in view, we get the expansion (4).

Let $\Gamma\subset D_{w_0}$ be a circle with centre at w_0, γ be a complex function analytic on the closed region $\mathbb{C}-(\Gamma\cup\operatorname{int}\Gamma)$ and $\gamma(\infty)=0$. For $w\in H_{w_0}$ we define

(6)
$$\varphi_\alpha(w)=(1/2\pi i)\int\limits_{\Gamma}\gamma(z)\,\Phi_\alpha(z,w)dz$$

and get easily that

(7)
$$\varphi_\alpha(w)=-(w/w_0)^{-1-\alpha}\int\limits_{l_{w_0}}\zeta^\alpha\gamma(\zeta)\exp(-w_0\zeta/w)d\zeta.$$

The function (6) is analytically continuable in the region $\mathbb{C}-l_{w_0}$. This continuation can be realised with the use of the representation (7), roughly speaking, by "rotation" of the ray l_{w_0} around the origin.

Further, if $w \in D_{w_0}$,

$$\varphi_\alpha(w) = \sum_{n=0}^{\infty} a_n^{(\alpha)}(\gamma)(w-w_0)^n, \tag{8}$$

where

$$a_n^{(\alpha)}(\gamma) = ((-1)^n/2\pi i w_0^n) \int_\Gamma \gamma(z) M_n^{(\alpha)}(z)\, dz,\ n=0,1,2,\ldots$$

If R is the radius of convergence of the power series (8), it is evident that $|w_0| \leq R \leq +\infty$. Suppose that $a_{n_k}^{(\alpha)}(\gamma)=0$ ($k=0,1,2,\ldots$) and that $R<+\infty$. In such a case the function $\varphi_\alpha(w)$ has only one singular point on the circle $|w-w_0|=R$. Moreover, this singular point is almost isolated and therefore is also good accessible [5, p. 28]. Then, according to a theorem of G. Polya. [5, p. 113] the (upper) density τ of the coefficients of the series (8), which are different from zero, is equal to one. But from the condition of the theorem it follows that $\tau \leq 1-\sigma < 1$. This contradiction shows that $R=+\infty$, i. e. φ_α is an entire function.

Putting $w=w_0 x (x>0)$ in (7), after some computation we get that $\varphi_\alpha(w_0 x) = -\int_0^\infty t^\alpha \gamma(xt) \exp(-t) dt$.

From the last equality it follows that $w_0^n \varphi_\alpha^{(n)}(0) = -\Gamma(n+\alpha+1)\gamma^{(n)}(0)$. But $\lim_{n\to+\infty} [|\varphi_\alpha^{(n)}(0)|(n!)^{-1}]^{1/n} = 0$, therefore $\lim_{n\to+\infty} [|\gamma^{(n)}(0)|(n!)^{-1}]^{1/n}$, i. e the function γ is analytic on the closed complex plane $\mathbb{C}$ and from the condition $\gamma(\infty)=0$ one can conclude that $\gamma \equiv 0$. Till now we proved that if L is a linear functional on the space $A(D_{w_0})$ and $L(M_{n_k}^{(\alpha)})=0$ $k=0,1,2,\ldots$, then $L=0$. This is sufficient for the system $\{M_{n_k}^{(\alpha)}(z)\}_{k=0}^\infty$ to be complete in $A(D_{w_0})$, while for every region $G \subset \mathbb{C}$ the space $A(G)$ is locally convex.

Theorem 2. *If $\alpha > -1$ and $\sigma = \lim_{k\to+\infty}(k/n_k) > 0$, the system $\{M_{n_k}^{(\alpha)}(z)\}_{k=0}^\infty$ is complete in the space $A(H)$, where H is the half-plane.* $\operatorname{Re} z < 0$.

Proof. For $z \in H$ and $\zeta \in \mathbb{C}-[0,+\infty)$ we put

$$\Psi_\alpha(z,\zeta) = -(-z)^\alpha \int_0^\infty (1+t(1-\exp\zeta))^{-1} t^\alpha \exp(zt)\, dt. \tag{9}$$

If $\zeta \in H$, then

$$\psi_\alpha(z,\zeta) = \sum_{n=0}^\infty M_n^{(\alpha)}(z) e^{n\zeta}. \tag{10}$$

Indeed, using Rodrigues formula [6, (5.1.5)] we get that

$$M_n^{(\alpha)}(z) = -\int_0^\infty (t-z)^{-n-1} t^{n+\alpha} \exp(-t)\, dt,\ n=0,1,2,\ldots \tag{11}$$

If $\operatorname{Re} z<0$, from (11) it follows that

$$M_n^{(\alpha)}(z) = -\int_{l_{-z}} (\zeta-z)^{-n-1} \zeta^{n+\alpha} \exp(-\zeta)\, d\zeta \tag{12}$$

$$= -(-z)^\alpha \int_0^\infty (1+t)^{-n-1} t^{n+\alpha} \exp(zt)\, dt$$

and the expansion (10) can be derived by means of the integral representation (12).

Let F be a linear functional on the space $A(H)$. It is not difficult to show that it has the form $F(f)=(1/2\pi i)\int_h (z-1)^{-1} w(z)f(z)dz$, where $h\subset H$ is a circle with centre at a point of the negative real semi-axis, ω is a complex function analytic on the closed region $\bar{\mathbb{C}}-\{h\cup \operatorname{int} h\}$ and $\omega(\infty)=0$.

If we define

$$\theta_a(\zeta)=(1/2\pi i)\int_h (z-1)^{-1}\omega(z)\,\Psi_a(z,\zeta)dz,$$

then for $\zeta\in H$ $\theta_a(\zeta)=\sum_{n=0}^{\infty} A_n^{(a)}(\omega)e^{n\zeta}$, where

$$A_n^{(a)}(\omega)=(1/2\pi i)\int_h (z-1)^{-1}\omega(z)\,M_n^{(a)}(z)dz=F(M_n^{(a)}).$$

Let $1\leqq R\leqq+\infty$ be the radius of convergence of the power series

(13) $$\psi_a(w)=\sum_{n=0}^{\infty}(-1)^n A_n^{(a)}(\omega)(w-1)^n.$$

It is not difficult to show that the function ψ_a is analytically continuable in the region $\mathbb{C}-[1,+\infty)$. Further, as in the proof of the Theorem 1' the assumption that $A_{n_k}^{(a)}(\omega)=0$ $(k=0,1,2,\ldots)$ and $R<+\infty$ leads to a contradiction. Therefore, if $F(M_{n_k}^{(a)})=0$, $k=0,1,2,\ldots$, the function ψ_a is analytic in the whole complex plane. For this function one can derive the following integral representation

(14) $$\psi_a(w)=\int_0^{\infty}\lambda(tw)t^a\exp(-t)dt,$$

where $\lambda(u)=[\omega(1)-\omega(u)](1-u)^{-1}$. Then, from (14) it follows that $\lambda^{(n)}(0)=\Psi_a(0)/\Gamma(n+a+1)$, i. e. λ is an entire function. It turns out that ω is analytic on $\bar{\mathbb{C}}$, therefore $\omega\equiv 0$. We proved that from the equalities $F(M_{n_k}^{(a)})=0$, $k=0,1,2,\ldots$, follows $F=0$, i. e. the system $\{M_{n_k}^{(a)}(z)\}_{k=0}^{\infty}$ is complete in $A(H)$.

Now we shall consider systems of the kind $\{G_{n_k}(z)\}_{k=0}^{\infty}$. In this case the method we use requires stronger condition about the density of the sequence $\{n_k\}_{k=0}^{\infty}$.

Theorem 3. *If* $\operatorname{Im} w_0\neq 0$ *and* $\lim_{k\to+\infty}(k/n_k)>3/4$, *the system* $\{G_{n_k}(z)\}_{k=0}^{\infty}$ *is complete in the space* $A(K_{w_0})$, *where* $K_{w_0}=\{w\in\mathbb{C}:|w-w_0|<|\operatorname{Im} w_0|\}$.

Proof. Using the generating function for the Hermite polynomials [6, (5.5.7)], we get easily that

(15) $$G(z,w)=\sum_{n=0}^{\infty}(n!)^{-1}G_n(z)w^n=-\exp(-w^2)\int_{-\infty}^{\infty}(t-z)^{-1}\exp(-t^2+2wt)\,dt.$$

Let $\varkappa\subset K_{w_0}$ be a circle with centre at the point w_0, $\delta(z)$ be a complex function analytic on the closed region $\bar{\mathbb{C}}-(\varkappa\cup\operatorname{int}\varkappa)$ and $\delta(\infty)=0$. If we define

(16) $$B(w)=(1/2\pi i)\int_{\varkappa}\delta(z)G(z,w)dz,$$

from (15) and (16) we get the representation

$$B(w)=-\exp(-w^2)\int_{-\infty}^{\infty}\delta(t)\exp(-t^2+2tw)dt.$$

It is evident that $B_w=\Sigma_{n=0}^{\infty}B_n(\delta)w^n$ is an entire function of order two and type one. Moreover, it is bounded on the real axis. According to a theorem of G. Polya [7, p. 625], the density of the coefficients $\{B_n(\delta)\}_{n=0}^{\infty}$, which are different from zero, is greater than 1/4.

Suppose that $\delta\not\equiv 0$, but nevertheless

(17) $$(1/2\pi i)\int_{\varkappa}\delta(z)G_{n_k}(z)dz=0,\ k=0,1,2,\ldots.$$

Then, having in view the condition of the theorem, for the same density we get that it must be smaller than 1/4. This contradiction shows that if the equalities (17) hold, then $\delta\equiv 0$, which means that the system $\{G_{n_k}(z)\}_{k=0}^{\infty}$ is complete in the space $A(K_{w_0})$.

In an analogous way one can prove the following

Theorem 4. *Let* $0\leqq\tau_0<+\infty$ *and* $H_{\tau_0}^{+}(H_{\tau_0}^{-})$ *denote the half-plane* $\operatorname{Im} z_0>\tau_0(\operatorname{Im} z_0<-\tau_0)$. *If* $\lim_{k\to+\infty}(k/n_k)>3/4$, *the system* $\{G_{n_k}(z)\}_{k=0}^{\infty}$ *is complete in the space* $A(H_{\tau_0}^{+})(A(H_{\tau_0}^{-}))$.

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