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HERMITE INTERPOLATING SPLINES OF SEVERAL VARIABLES

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The aim of this paper is to give a construction of Hermite interpolating splines of d-variables, i. e. splines which interpolate functions of d-variables with their derivatives up to n-th order at given points. These splines are polynomials on simplexes of a given triangulation. For the case of two variables this problem was considered in [1]. We also construct (cf. Section 3) an operator from a space of functions defined on some finite set of points into a space of splines*, similar to the operator L described by W. Riabenkij [2, 4].

Let K be a compact domain in R^d, i. e. $K=\mathrm{cl}(\mathrm{int}\, K)$, K is connected. $C^n(K)$ denotes the space of n-times continuously differentiable functions on K considered with norm $\|f\|_n=\max\{\|D^k f\|: k=0,\dots,n\}$, where $\|D^k f\| =\sup\{\|D^k f(x)\|: x\in K\}$ and $\|D^k f(x)\|$ is the norm of a k-linear operator $D^k f(x)$. If D is a multilinear operator, then we write $D(\underbrace{x,\dots,x}_{n}, \underbrace{y,\dots,y}_{k}, \dots, \underbrace{z,\dots,z}_{l})=D(x^n, y^k,\dots,z^l)$.

$\omega^k(f,t)=\sup\{\|D^k f(x)-D^k f(y)\|: x,y\in K;\ x\neq y\}$ is the k-th modulus of continuity of $f\in C^k(K)$.

If A is a finite set, then $C(A)$ is the space of all real functions defined on A. P^d_n denotes the space of all polynomials of d-variables, of degree not greater than n. If $\alpha=(\alpha_1\dots,\alpha_d)\in N^d$ is a multi-index, then $|\alpha|=\alpha_1+\dots+\alpha_d$ and $\|\alpha\|=\max\{\alpha_1,\dots,\alpha_d\}$. $|x_1-x_2|$ denotes the Euclidean distance in R^d, where $x_i=(x_i(l), \dots x_i(d))$; $i=1,2$. $(\bar e)^d_{i=1}$ is the canonical basis in R^d, $\Gamma=\mathrm{conv}\{0,\bar e_1,\dots,\bar e_d\}$. Simplexes in R^d will be denoted by Δ; $\varrho(\Delta)$ and $\delta(\Delta)$ denote a radius of the inscribed ball in Δ and a diameter of Δ, respectively.

$$T(s)=\{\Delta\subset R^d: \delta(\Delta)\leq s\,.\,\varrho(\Delta)\}.$$

If T is a set of simplexes, then $\delta(T)=\sup\{\delta(\Delta):\Delta\in T\}$ and T^0 is the set of all vertices of simplexes from T.

For a compact domain K we define a measure of unconvexity $v(K)$ by

$$v(K)=\sup\{\varrho_K(x,y)\,.\,|x-y|^{-1}: x,y\in K;\ x\neq y\},$$

* The complete proofs will be published in *Studia Mathematica.*

where $\varrho_K(x, y)$ is a length of the shortest arc l such that $x, y \in l \subset K$. If $v(K) < +\infty$, then we call K a domain of a finite unconvexity. We say that K is a domain of a finite taperingity if there exist constants s, h and a set of simplexes T such that K is a sum of simplexes from T and $T \subset \{\Delta \subset R^d : \delta(\Delta) \geq h\} \cap T(s)$.

1. Let $e_0, \ldots, e_d$ be vertices of a simplex $\Delta \subset R^d$. We denote

$$(1.1) \qquad e_{ij} = (e_{ij}(\Delta) =) = \begin{cases} e_i - e_{j-1} & i \geq j,\ i = 0, \ldots, d, \\ e_i - e_j, & i < j,\ j = 1, \ldots, d \end{cases}$$

and

$$(1.2) \quad \mathfrak{Q}_n(\Delta) \{ f \in C^{2n+2}(\Delta) : D^{2n+2} f(x) (e_{ij}^{2n+2}) = 0 \text{ for } x \in \Delta;\ i = 0, \ldots, d;\ j = 1, \ldots, d \}$$

Obviously functions from $\mathfrak{Q}_n^{(\Delta)}$ are polynomials.

Proposition 1.1. $\dim \mathfrak{Q}_n(\Delta) = (d + l)(n + l)^d$.
More precisely for each real matrix

$$\mathfrak{A} = [a_{\alpha i}]_{i = 0, \ldots, d;\ \|\alpha\| \leq n}$$

there exists a unique polynomial $Q_{\mathfrak{A}} \in \mathfrak{Q}_n(\Delta)$ such that

$$(1.3) \qquad D^{|\alpha|} Q_{\mathfrak{A}}(e_i)(e_{i1}^{\alpha_1}, \ldots, e_{id}^{\alpha_d}) = a_{\alpha i} \text{ for } i = 0, \ldots, d;\ \|\alpha\| \leq n.$$

Moreover if $a_{\alpha i} = 0$ for $i = 0, \ldots, m$; $\|\alpha\| \leq n$, then

$$(1.4) \qquad D^k Q_{\mathfrak{A}}(x) = 0 \text{ for } k = 0, \ldots, n;\ x \in \operatorname{conv}\{e_0, \ldots, e_m\}.$$

Sketch of a proof. Observe that $\dim \mathfrak{Q}_n(\Delta) \leq (d+l)(n+l)^d$ because it is so for $\Delta = \Gamma$; the following polynomials belong to $\mathfrak{Q}_n(\Gamma)$ and are linearly independent $P_{i\alpha}(x) = x(i)^{n+1} \cdot x(0)^{\alpha_1} \cdot \ldots x(i-1)^{\alpha_i} x(i)^{\alpha_{i+1}} \cdot \ldots \cdot x(d)^{\alpha_d}, i = 0, \ldots, d$ $\|\alpha\| \leq n$, where $x(0) = 1 - x(1) - \ldots - x(d)$. The condition (1.4) we can prove by induction with respect to d and k. It follows from (1.4) (for $m = d$) that $\dim \mathfrak{Q}_n(\Delta) = (d+l)(n+l)$.

Proposition 1.2. *Let $s \in R_+$, $m \in N$, $m \leq n$. There exists a constant $c = c(n, d, s)$ such that if $\Delta \in T(s)$, $f \in C^m(\Delta)$, $Q \in \mathfrak{Q}_n(\Delta)$ and*

$$(1.5) \quad D^{|\alpha|} Q(e_i)(e_{i1}^{\alpha_1}, \ldots, e_{id}^{\alpha_d}) = \begin{cases} D^{|\alpha|} f(e_i)(e_{i1}^{\alpha_1}, \ldots, e_{id}^{\alpha_d}), & |\alpha| \leq m; \\ 0 \qquad , & |\alpha| > m;\ \|\alpha\| \leq n, \end{cases} \quad i = 0, \ldots, d,$$

then

$$(1.6) \quad \| D^{k+1} Q \| \leq c \max \{ \omega^l(f, \delta(\Delta)) (\delta(\Delta))^{l-k-1} : l = k, \ldots, m \},\ k = 0, \ldots, m.$$

Proposition 1.2 we can prove, for $\Delta = \Gamma$ first, by induction with respect to k, beginning from k equal to m.

2. Let $W \subset R^d$ be a polyhedron and T be its triangulation.

Definition 2.1. *We define a space of splines $\mathfrak{S}_n(T) \subset C^n(W)$ as a set of all functions whose restrictions to any simplex $\Delta \in T$ belong to $\mathfrak{Q}_k(\Delta)$, i. e.*

$$(2.1) \qquad \mathfrak{S}_n(T) = \{ g \in C^n(W) : g_{|\Delta} \in \mathfrak{Q}_n(\Delta) \text{ for } \Delta \in T \}.$$

We see from Proposition 1.1 that these splines are Hermite interpolating splines, i. e. for each $f \in C^m(W)$ there exists $g \mathfrak{S}_n(T)$ such that

$$(2.1) \qquad D^k f(v) = D^k g(v) \text{ for } v \in T^0;\ k = 0. \ldots, \min(n, m).$$

If we compare (2.2) with (1.3) we see that for $m<nd$ derivatives of order greater than m in certain directions are undefined. If we choose them to be zero, the resulting interpolating spline is uniquely determined. We denote it by $S_T^{m,n} f$.

Corollary 2.2. *Let $s \in R_+$, $m \leq n$. Then there exists a constant $c=c(n, d, s)$ such that for each triangulation T with*

$$T \subset T(s) \text{ and } s \,.\, \min\{\delta(\Delta) : \Delta \in T\} \geq \delta(T) \tag{2.3}$$

we have the estimate

$$\| D^k(S_T^{m,n} f - f) \| \leq c \,.\, \max\{\omega^l(f, \delta(T))(\delta(T))^{l-k} : l=k, \ldots, m\},\ k=0, \ldots, m \tag{2.4}$$

This Corollary is a simple consequence of Proposition 1.2.

Remark 2.3. If we have a sequence of triangulations $(T_j)_{j=0}^{\infty}$ of W such that $\lim_{j\to\infty} \delta(T_j)=0$ and all T_j satisfy (2.3) for some s, then $\lim_{j\to\infty} \| S_{T_j}^{m,n} f - f \|_m = 0$ for each $f \in C^m(W)$ $(m \leq n)$.

3 Let W and T be as in Section 2. In this section we will construct an operator from $C(V)$ into $\mathfrak{S}_n(T)$, where V is some subset of T.

Definition 3.1. *Let a be greater than one. We define $\mathfrak{N}_n^d(a)$ as a family of all finite subsets x of R^d for which there exists a sequence of subsets $x_0 \subset x_1 \subset \cdots \subset x_n = x$ satisfying the following conditions*

$$\operatorname{card} x_k = \dim \mathfrak{P}_k^d,\ k=0, \ldots, n \tag{3.1}$$

$$\max\{\| D^j P(X_0)\| \operatorname{diam} x)^j : j=0, \ldots, k\} \leq a \max\{|P(x)| : x \in x_k\} P \in \mathfrak{P}_k^d; \quad k=0, \ldots, n, \tag{3.2}$$

where x_0 is a unique element of x_0. (We write in such a situation $x=x(x_0)$.)

It is easy to see that if $P \in \mathfrak{P}_n^d$ and $P_{|x}=0$, then $P=0$.

Lemma 3.2. *Let $x=x(x_0) \in \mathfrak{N}_n^d(a)$, $f \in C^n(\operatorname{conv} x)$ and $f_{|x}=0$. Then*

$$\|D^k f(x_0)\| \leq (4a)^{n-1} \omega^n(f, \operatorname{diam} x)(\operatorname{diam} x)^{n-k}\ \ k=0, \ldots, n. \tag{3.3}$$

Let V be a subset of T^0. We assume that there exists a retraction $\pi: T^0 \to V$ such that $|\pi(x)-x| \leq b \,.\, \delta(T)$. We assume also that for any $v \in V$ we can find $x_v = x(v) \in \mathfrak{N}_n^d(a)$ such that $x_v \subset V$, $\operatorname{diam} x_v \leq b \,.\, \delta(T)$. Let $\varphi \in C(V)$. For each $v \in V$ we define $P_v \in P_n^d$ such that $P_{v|x_v} = \varphi_{|x_v}$. Such P_v is unique. Let $g \in C^n(W)$ be such that

$$D^k g(x) = D^k P_{\pi(x)}(x) \text{ for } x \in T^0;\ k=0, \ldots, n.$$

Now we define a spline $L_T^n \varphi$ by $L_T^n \varphi = S_T^{nn} g$. Observe that $L_T^n \varphi$ does not depend on the choice of g.

Proposition 3.3. *There exists a constant $c=c(n, d, s, a, b)$ such that for $f \in C^m(W)$ $(m \leq n)$ we have $\| D^{k+1} L_T^n f \| \leq c \,.\, \omega^k(f, \delta(T))(\delta(T))^{-1}$, $k=0, \ldots, m$,* where $L_T^n f = L_T^n(f_{|V})$.

Let us note that the spline $L_T^n f$ is a local spline, i.e. a value of $L_T^n f(x)$ depends only on values of f in some neighbourhood of x.

Let us suppose that K is a domain of a finite unconvexity and a finite taperingity and let h be a sufficiently small positive number. Let V be a h-net in K such that $\min\{|x-y|: x, y\in V;\ x\neq y\}\geq v.h$. We can construct a polyhedron $W\supset K$ and its triangulation T such that $T^0\cap K=V$; $T\subset T(s)$; $\delta(T)\leq s.h$; $\min\{\delta(\Delta):\Delta\in T\}\geq s^{-1}h$ for some $s=s(d, n, K, r)$. Now we define $L_V: C(K)\to C^n(K)$ by

$$L_Vf=(L_T^n(f_{|V}))_{|K}.$$

Theorem 3.4. *There exists a constant $c=c(n, d, K, r)$ such that for $m\leq n$*

(3.4) $\qquad \|D^kL_Vf\|\leq c\|D^kf\| \qquad k=0, \ldots, m;\ f\in C^m(K),$

(3.5) $\qquad \omega^k(L_Vf, t)\leq ct\omega^k(f.h)h^{-1} \qquad k=0, \ldots, m;\ f\in C^m(K),$

(3.6) $\qquad$ *If $f, g\in C(K)$ and $f_{|V}=g_{|V}$, then $L_Vf=L_Vg$ and $L_Vf_{|V}=f_{|V}$.*

The proof follows from Proposition 3.3 and Lemma 3.2.

4. We can generalize Theorem 3.4 replacing a domain K by a differentiable compact manifold M which can have a boundary and certain other singularities. We can obtain the operator L_V using the suitable partition of unity on the manifold M.

We can use splines L_Vf to construct the Schauder bases in spaces of differentiable functions.

Theorem 4.1. *Let K be as in Theorem 3.4, let A be a dense, countable subset of K and let n be a natural number. Then there exists a sequence of functions $(\varphi_i)_{i=1}^{\infty}$ which is a basis in each space $C^m(K)$ $(m=0, \ldots, n)$ and is an interpolating basis in $C(K)$ with A as a set of nodes.*

A construction of functions φ_i is similar to the one described in [4]. We must divide the set A into a sequence of suitable subsets $(V_j)_{j=1}^{\infty}$ and apply the operators L_Vk (where $V^k=\bigcup_{j=1}^{k}V_j$), to functions which are equal to one in some point of V_k and are equal to zero in other points of V^k.

For the special sets we consider Theorem 4.1 as an improvement of a result of W. Gurarij [3].

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