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APPROXIMATION WITH BELL-SHAPED FUNCTIONS

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Summary. If $f(t)$ is bounded and measurable on the whole of the real line and $\beta(t$ is a "bell-shaped" function, the approximation of $f(x)$ by

$$U_\varrho(f;\,x)=a_\varrho \int_{-\infty}^{\infty} f(x-t)\beta^\varrho(t)dt$$

is considered under very general conditions concerning $\beta(t)$ if $\varrho\to\infty$.

If $f''(x)$ exists, a Voronovskaya type formula of the form

$$\varrho^\lambda\{U_\varrho(f;\,x)-f(x)\}\ \ A_\varrho f'(x)+B_\varrho f''(x)+o(1)\quad(\varrho\to\infty)$$

is discussed. Here at least one of the coefficients A_ϱ and B_ϱ is $O(1)$, the other one is either $O(1)$ or $o(1)$.

Applications are given.

1. In 1959 P. Korovkin [2] studied a minor modification of the following operators $K_n\colon C[a,\,b]\to C[a,\,b]$; $n=1,\,2,\dots$, where $[a,\,b]$ is a bounded, closed interval of the real axis R, $a<0<b$,

$$K_n(f(t);\,x)=I_n^{-1}\int_a^b f(x-t)\beta^n(t)dt,$$

with $f\in C[a,\,b]$, $x\in[a,\,b]$, $\beta(t)\in C[-\gamma,\,\gamma]$, $0<b-a\leqq\gamma<\infty$ and

$$I_n=\int_{-\gamma}^{\gamma}\beta^n(t)dt.$$

In his considerations $\beta(t)$ possesses moreover the properties, that $\beta(0)=1$, $0\leqq\beta(t)<1$ if $0<|t|\leqq\gamma$. Korovkin proved that for every fixed $x\in(a,\,b)$ the approximation property

(1) $$K_n(f;\,x)\to f(x)\quad(n\to\infty)$$

holds.

However, Korovkin did not occupy himself with the speed with which the approximation (1) takes place for the general case of $\beta(t)$ satisfying the above conditions. In 1973 R. Bojanic and O. Shisha [1] derived an inequality for the difference $K_n(f;\,x)-f(x)$ if $x\in[a+\delta,\,b-\delta]\,(0<\delta<(b-a)/2)$, which involved the modulus of continuity of f on $[a, b]$, but only under more stringent conditions on $\beta(t)$ than Korovkin's: $\beta(t)$ should be even on $[-\gamma, \gamma]$ and monotonically decreasing on $[0, \gamma]$. They noticed that for the derivation of such an inequality it was essential to know more about the asymptotic behaviour of $1-\beta(t)$ as $t \downarrow 0$.

In this paper we considerably generalize the Korovkin theory and, with respect to the speed with which the image of f tends to $f(x)$ if the parameter tends to infinity, we suppose that $f''(x)$ exists and we give an approximation formula of Voronovskaya type.

A generalization of the results of Bojanic and Shisha will be published in a subsequent paper.

2. It is supposed that the real-valued function $\beta(t)$ is defined on the whole of the real axis R and possesses the following four properties:

1. $\beta(t) \geqq 0$ on R,
2. $\beta(t)$ is continuous at $t=0$ and $\beta(0)=1$,
3. for all $\delta>0: \sup\{\beta(t)<1: |t| \geqq \delta\}$,
4. $\beta(t) \in L_1$, i. e. $\int_{-\infty}^{\infty} \beta(t)dt$ exists in the sense of Lebesgue.

Such a function $\beta(t)$ is called a bell-shaped function. The class of all bell-shaped functions is denoted by B.

Let M be the class of all functions $f(t)$ that are defined, bounded and Lebesgue-measurable on the whole of R.

Then, for all positive ϱ, $\varrho \geqq 1$, the operators U_ϱ are defined on M by

$$U_\varrho(f(t); x) = I_\varrho^{-1} \int_{-\infty}^{\infty} f(x-t)\beta^\varrho(t)dt,$$

where $\beta \in B$, $x \in R$ and $I_\varrho = \int_{-\infty}^{\infty} \beta^\varrho(t)dt$.

3. Theorem 1 (approximation theorem). *If $\beta \in B$, $f \in M$ and $x \in R$ is a point of continuity of f, then $U_\varrho(f; x) \to f(x)$ if $\varrho \to \infty$.*

Sketch of proof. Since $f(t)$ is continuous at $t=x$, there exists to each $\varepsilon>0$ a $\delta>0$, such that for all t with $|t|<\delta$ $|f(x-t)-f(x)|<\varepsilon/2$. By property 3. of β $\inf\{(1-\beta(t)): |t| \geqq \delta\}>0$ and, consequently, there exists a constant $P>0$ such that for all t with $|t| \geqq \delta$ $|f(x-t)-f(x)| \leqq P(1-\beta(t))$. Hence, for all $t \in R$

$$|f(x-t)-f(x)| < \varepsilon/2 + P(1-\beta(t)).$$

Then

$$(2) \quad |U_\varrho(f; x)-f(x)| < \varepsilon/2 + (P/I_\varrho) \int_{-\infty}^{\infty} (1-\beta(t))\beta^\varrho(t)dt = \varepsilon/2 + P\{1 - I_{\varrho+1}/I_\varrho\}.$$

Because [3, lemma 4] $(I_{\varrho+1}/I_\varrho) \to 1$ $(\varrho \to \infty)$ it follows from (2) that for all sufficiently large ϱ the inequality

$$|U_\varrho(f; x)-f(x)| < \varepsilon$$

holds. This proves the theorem.

4. It will now be supposed that $\beta(t) \in B$ and possesses, in addition, property

5. $\begin{cases}\beta(t)=1-ct^{\alpha}+\varphi(t), \text{ if } t>0, \text{ with } c>0,\ \alpha>0,\ \varphi(t)=o(t^{\alpha})\ (t\downarrow 0),\\ \beta(t)=1-c'\,|t|^{\alpha'}+\psi(t), \text{ if } t<0, \text{ with } c'>0,\ \alpha'>0,\ \psi(t)=o(|t|^{\alpha'})\ (t\uparrow 0).\end{cases}$

Theorem 2. *If $\beta\in B$ possesses property 5. with $\alpha>\alpha'$, if $f\in M$ and if at a point $t=x$ $f''(x)$ exists, then*

$$\varrho^{1/\alpha}\{U_\varrho(f;\,x)-f(x)\}=-c^{-1/\alpha}(\Gamma(2/\alpha)/\Gamma(1/\alpha))f'(x)+o(1)\quad(\varrho\to\infty);$$

if $\alpha'>\alpha$ (instead of $\alpha>\alpha'$), then

$$\varrho^{1/\alpha'}\{U_\varrho(f;\,x)-f(x)\}=c^{-1/\alpha'}(\Gamma(2/\alpha')/\Gamma(1/\alpha'))f'(x)+o(1)\quad(\varrho\to\infty).$$

Theorem 3. *If $\beta\in B$ possesses property 5. with $\alpha=\alpha'$, if $f\in M$ and if at a point $t=x$ $f''(x)$ exists, then, if $c\neq c'$*

$$\varrho^{1/\alpha}\{U_\varrho(f;\,x)-f(x)\}=(c'^{1/\alpha}-c^{1/\alpha})\Gamma(2/\alpha)/(cc')^{1/\alpha}\Gamma(1/\alpha)\cdot f'(x)+o(1)\ (\varrho\to\infty);$$

and if $c=c'$,

$$\varrho^{1/\alpha}\{U_\varrho(f;\,x)-f(x)\}=o(1)\quad(\varrho\to\infty). \tag{3}$$

5. From the latter part of theorem 3 it follows that if in property 5. $\alpha=\alpha'$ and $c=c'$, it is required that more is known about $\varphi(t)$ $(t\downarrow 0)$ and $\psi(t)$ $(t\uparrow 0)$ in order to be able to say something more precisely than only $o(1)$ in (3). In this connection the following theorem serves as an example.

Theorem 4. *If $\beta\in B$ and β possesses property 5. in the strenghthened form*

5'. $\begin{cases}\beta(t)=1-ct^{\alpha}+dt^{\mu}+\varphi(t), \text{ with } c>0,\ \alpha>0,\ \mu>\alpha,\ \varphi(t)=o(t^{\mu})\ (t\downarrow 0),\\ \beta(t)=1-c\,|t|^{\alpha}+d'\,|t|^{\mu}+\psi(t), \text{ with } d'\neq d,\ \psi(t)=o(|t|^{\mu})\ (t\uparrow 0),\end{cases}$

if $f\in M$ and if at a point $t=x$ $f''(x)$ exists, then $\varrho^{\sigma/\alpha}\{U_\varrho(f;x)-f(x)\}=p(x)+o(1)$ $(\varrho\to\infty)$, where $\sigma=\min(\mu-\alpha+1,\,2)$ and

(i) *if $\alpha<\mu<\alpha+1$, then $\sigma=\mu-\alpha+1$ and $p(x)=2^{-1}c^{-(\mu+1)/\alpha}(d'-d)(\Gamma((\mu+2)/\alpha)/\Gamma(1/\alpha))f'(x)$;*

(ii) *if $\mu=\alpha+1$, then $\sigma=2$ and $p(x)=2^{-1}c^{-2/\alpha}\Gamma(3/\alpha)\{-(3d/\alpha c)f'(x)+f''(x)\}/\Gamma(1/\alpha)$;*

(iii) *if $\mu>\alpha+1$, then $\sigma=2$ and $p(x)=2^{-1}c^{-2/\alpha}(\Gamma(3/\alpha)/\Gamma(1/\alpha))f''(x)$.*

If in property 5'

(4) *$d'=d$ and for an $\eta>0:\varphi(t)=\psi(-t)$ $(0\leq t\leq\eta)$,*

then $$\varrho^{2/\alpha}\{U_\varrho(f;\,x)-f(x)\}=2^{-1}c^{-2/\alpha}(\Gamma(3/\alpha)/\Gamma(1/\alpha))f''(x)+o(1)\ (\varrho\to\infty).$$

Remarks. 1. If (4) holds, $\beta(t)$ is even on the interval $-\eta\leq t\leq\eta$.

2. From theorem 4 it follows that if μ $(\mu>\alpha)$ increases and passes $\alpha+1$, in $p(x)$ the term involving $f''(x)$ comes up, while the term involving $f'(x)$ disappears. In fact, then the influence of t^{μ} on t^{α} is diminishing in a small neighbourhood of $t=0$ and $\beta(t)$ resembles more and more an even function in this small neighbourhood.

For full proofs of the above theorems and many applications the reader is referred to [3].

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