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ON THE CONVERGENCE OF PADÉ APPROXIMANTS TO HAMBURGER FUNCTIONS

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Summary. In this paper we consider sequences of Padé approximants $[m_i/n_i](z)$ ($m_i, n_i \in \mathbb{N}$ $i=1, 2, \ldots$) to functions $f(z)=\int_I (x-z)^{-1}\, d\mu(x)$, where μ is a positive measure, and $I \subseteq R$ a compact interval. The following theorem is proved:

If $m_i+n_i \to \infty$, for $i \to \infty$, $m_i-n_i \in 2\mathbb{Z}$ for $i=1, 2, \ldots$ and

$$\limsup_{i\to\infty} \left(|m_i-n_i|/(m_i+n_i) \right)=\lambda<1,$$

then $\lim_{i\to\infty} [m_i/n_i](z)=f(z)$ locally uniform in $\{z \in \mathbb{C} : |\psi(z)|^{1-\lambda} |z/R|^{\lambda}>1\}$, where ψ is the Riemann mapping function of the exterior of I of $\{|z|>1\}$, with $\psi(\infty)=\infty$, and $R=\max\{|x| \mid x \in I\}$.

1. Introduction. In this paper we prove a theorem on the convergence of Padé approximants $[m/n]$ ($m, n \in \mathbb{N}$) to *Hamburger functions*, i. e. functions of the form

(1) $$f(z)=\int_I (x-z)^{-1}\, d\mu(x),$$

where μ is a positive measure on $\mathbb{R}$. We suppose that $I \subseteq \mathbb{R}$ is a compact interval. First some definitions (for the theory of Padé approximation the reader is referred to O. Perron [3, Chap. 10], H. Wall [5, Chap. 20], or G. Baker [1]): The Padé approximant $[m/n](z)$ to function (1) is defined, as the rational function

(2) $$[m/n]\,(z)=p_{m,n}(1/z)/q_{m,n}(1/z)$$

where $p_{m,n}$ and $q_{m,n}$ are polynomials of degree $\leq m$ and $\leq n$, respectively satisfying

(3) $$f(z)\, q_{m,n}(1/z)-p_{m,n}(1/z)=O(z^{-m-n-1}),$$

H. Padé [2]. The function $[m/n]$ is uniquely defined by (2) and (3). The doubly infinite array

(4) $$([m/n])_{m,n=0,\,1,\,2,\,\ldots}$$

is called the *Padé table* of the function $f(z)$. Its first row contains the partial sums of the power series expansion of $f(z)$ at infinity. The diagonal $\{[n/n](z);\ n=0, 1, 2, \dots\}$ is known as the associated continued fraction to the power series of $f(z)$ O. Perron [3, §77 III]. A. Markov gave a short and elegant proof of the fact that this continued fraction converges to $f(z)$ for all $z \notin I$. For general sequences $\{[m/n]\}$ of the Padé table a convergence result has been proved in P. Wynn [6] for Stieltjes functions, i. e. for functions of type (1) with I contained in the positive or negative half-axis. For Hamburger functions the convergence is proved for $z \in \mathbb{R}$ with $|z| > \max\{|x| \mid x \in I\}$ in P. Wynn, [7, Th. 6]. Also for Stieltjes functions G. A. Baker Jr. states a convergence theorem [1, Th. 16. 2], but it contains an inaccuracy such that the given domain of convergence is too large, this can easily be seen by looking at the sequence $\{[0/0], [1/0], [2/0] \dots\}$. The domain of convergence proved here is an improvement of the result of P. Wynn.

2. The Result. Let ψ denote the Riemann mapping function of the exteriour of I on $\{|z|>1\}$ with $\psi(\infty)=\infty$, and let $R=\max\{|x| : x \in I\}$. We are concerned with sequences $\{[m_\nu/n_\nu]\}$ of the Padé table of function (1) with $m_\nu, n_\nu \in \mathbb{N}$ satisfying

$$\lim_{\nu\to\infty}(m_\nu+n_\nu)=\infty:\ m_\nu-n_\nu \in 2\mathbb{Z},\ \nu=1, 2, \dots . \tag{5}$$

We set

$$\lambda = \limsup_{\nu\to\infty} |m_\nu-n_\nu|/(m_\nu+n_\nu). \tag{6}$$

Theorem. *If the sequence* $\{(m_\nu, n_\nu)\}$ *satisfies* (5), *then we have*

$$\limsup_{\nu\to\infty} |f(z)-[m_\nu/n_\nu](z)|^{1/(m_\nu+n_\nu)} \le |\psi(z)|^{\lambda-1}|R/z|^{\lambda} \tag{7}$$

locally uniform in $D_\lambda=\{z \in \mathbb{C}:\ |\psi(z)|^{\lambda-1}|R/z|^{\lambda}<1\}$.

Remarks: 1) For sequences with $m_\nu \ge n_\nu$ the inequality (8) holds for all $z \notin I$.

2) If $\lambda=0$, i. e. if $n_\nu \to \infty$ and $(m_\nu-n_\nu)/n_\nu \to 0$, then $[m_\nu/n_\nu](z)$ converges locally uniformly to $f(z)$ for all $z \notin I$. This extends the theorem of Markov.

3) If $\lambda>0$, then D_λ contains inner points, and is growing monotonously with increasing λ. At last, when $\lambda=1$, we have $D_1=\{|z|>R\}$, which corresponds to the circle of convergence of the power series expansion of $f(z)$.

3. The Proof. For $k \in 2\mathbb{N}$ we can develop $f(z)$ as

$$\begin{aligned} f(z) &= a_0 z^{-1}+a_1 z^{-2}+\dots+a_{k-1} z^{-k}+z^{-k}\int (x-z)^{-1}x^k\,d\mu(x) \\ &= P_k(1/z)+z^{-k}f_k(z), \end{aligned} \tag{8}$$

where $f_k(z)$ denotes the integral in the first line of (8), it is again a function of type (1). P_k is a polynomial of degree $\le k$. From (3) it follows that

$$\begin{aligned} f(z)q_{n+k,n}(1/z)-p_{n+k,n}(1/z) &= O(z^{-2n-k-1}) \\ &= z^{-k}f_k(z)q_{n+k,n}(1/z)-\{p_{n+k,n}(1/z)-q_{n+k,n}(1/z)P_k(1/z)\}. \end{aligned} \tag{9}$$

The expression in brackets on the last line is a polynomial in z^{-1} of degree $\le n+k$, and it contains the factor z^{-k}. Hence, we get by multiplying with z^k

$$(10)\quad f_k(z)\, q_{n+k,n}(1/z) - z^k \{p_{n+k,n}(1/z) - q_{n+k,n}(1/z)\, P_k(1/z)\} = O(z^{-2n-1}).$$

Since now the second term of the left-hand side is a polynomial in $1/z$ of degree $\leq n$, it follows from (3) that it is the numerator of the Padé approximant $[n/n]_{f_k}(z)$ of the function $f_k(z)$, and $q_{n+k,n}$ is its denominator. From this it follows that $[m/n]_{f_k}(z) = z^k\{[n+k/n]_f(z) - P_k(1/z)\}$, and thus that

$$(11)\quad [n+k/n]_f(z) = a_0 z^{-1} + a_1 z^{-2} + \ldots + a_{k-1} z^{-k} + z^{-k}[n/n]_{f_k}(z),$$

where $[n+k/n]_f$ is the Padé approximant to $f(z)$.

With (8) and (11) we can reduce the convergence problem of general sequences to that of a diagonal sequence $\{[n/n]_{f_k}\}$, but now to a function $f_k(z)$ that changes with k. We have

$$(12)\quad f(z) - [n+k/n]_f(z) = z^{-k}\{f_k(z) - [n/n]_{f_k}(z)\}.$$

Let $C_r\,(r>1)$ denote the set $\{z \in \mathbb{C} : |\psi(z)| = r\}$. Since k is supposed to be even, we have

$$(13)\quad |f_k(z)| = |\int (x-z)^{-1} x^k\, d\mu(x)| \leq R^k \mu(I)/\operatorname{dist}(C_r, I)$$

for all $z \in C_r$. Since $f_k(z)$ is a Hamburger function, the Padé approximant $[n/n]_{f_k}$ can be represented as

$$(14)\quad [n/n]_{f_k}(z) = \sum_{\nu=1}^{n} a_{\nu,n}^{(k)}/(z - \gamma_{\nu,n}^{(k)}),$$

where all $\gamma_{\nu,n}^{(k)}\,(\nu = 1, \ldots, n)$ lie on I, and $a_{\nu,n}^{(k)} < 0$ for $\nu = 1, \ldots, n$ with $\Sigma_{\nu=1}^{n} - a_{\nu,n}^{(k)} = \int x^k\, d\mu(x) \leq R^k \mu(I)$ O. Perron [3, § 69, Theorem 4]. From (14) it follows that

$$(15)\quad |[n/n] f_k(z)| \leq R^k \mu(I)/\operatorname{dist}(C_r, I).$$

(12, (13) and (15) give together for $z \in \operatorname{Ext}(C_r)$ that

$$(16)\quad |\psi(z)^{2n} (z/R)^k \{f(z) - [n+k/n]_f(z)\}| \leq 2r^{2n} \mu(I)/\operatorname{dist}(C_r, I),$$

and from this we get with $m = n+k$ that

$$(17)\quad |f(z) - [m/n](z)|^{1/(m+n)} \leq (2r^{2n})^{1/(m+n)} |\psi(z)|^{-2n/(m+n)} |R/z|^{(m-n)/(m+n)}.$$

Since $|\psi(z)| \geq |z/R|$ for all $z \notin I$, the right-hand side of (17) increases mo notonic with $(m-n)/(m+n)$. Hence, (7) follows from (17) and (6) for sequences $\{[m_\nu/n_\nu]\}$ with $m_\nu \geq n_\nu$.

We will reduce the case of sequences $\{[m_\nu/n_\nu]\}$ with $m_\nu < n_\nu$ to the former one, using that

$$(18)\quad f(z)^{-1} = \mu(I)^{-1} z - \mu(I)^{-2} \int x\, d\mu(x) - \int (x-z)^{-1} d\tilde{\mu}(x) = b_1 z + b_0 - \tilde{f}(z)$$

and for $m, n \geq 1$ that

$$(19)\quad [m/n]_f(z)^{-1} = b_1 z + b_0 - [n-1/m-1]_{\tilde{f}}(z),$$

where $\tilde{\mu}$ is a positive measure on I. Thus, $\tilde{f}(z)$ is also of type (1). Formula (18) is a cousequence of the fact that functions of type (1) have a positive definite power series expansion and that this property is preserved by taking the reciprocal (H. Wall [4. Theorem 8]). Formula (19) follows easily from definition (3). With (18) and (19) we get

$$
(20) \qquad |f(z)-[m/n]_f(z)| = \frac{|f(z)^{-1}-[m/n]_f(z)^{-1}|}{|f(z)|^{-1}|[m/n]_f(z)|^{-1}} \leq \frac{|\tilde{f}(z)-[n-1/m-1]_{\tilde{f}}(z)|}{|f(z)|^{-1}(|f(z)|^{-1}-|\tilde{f}(z)-[n-1/m-1]_{\tilde{f}}(z)|)}.
$$

Since $f(z) \neq 0$ for all $z \notin I$, (7) follows for sequences $\{(m_\nu/n_\nu]\}$ with $m_\nu < n^\nu$ from (20) and the first part of this proof. q. e. d.

The assertion of the theorem can further be improved, if inequality (16) is multiplied by the modulus of a function, analytic in the complement of the support of the measure μ, with boundary values on supp (μ) such that (16) becomes sharper than that given above, for $z \in C_r$ and $r \to 1$. Such a function can be defined with the help of a special logarithmic potential. The development of these tools and the modified proof is too long to be given here.

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