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ON CONSTRAINED POLYNOMIAL APPROXIMATION

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Summary. The monotone approximation (i. e. approximation of functions with non-negative first differences by monotone increasing polynomials) has a well-developed theory. We give a two-fold generalization of the results of this kind, by approximating functions with higher order non-negative differences by polynomials of the same type, in terms of the modulus of smoothness.

In the last twelve years the problem of approximation of monotone continuous functions by monotone polynomials has been widely investigated. Quantitative results for the degree of approximation as well as generalizations to the so-called co-monotone approximation were obtained.

The object of this paper is to present a result of this type. While monotonicity and convexity means that the first and second differences of the function considered are non-negative, respectively, we shall consider functions for which higher order differences will be of constant sign, and the error-estimates will be obtained in terms of the modulus of smoothness, too, besides the modulus of continuity. Similar problems (for finitely differentiable functions) have been considered by J. A. Roulier [1] (cf. also L. Raymon [3]).

Let $f(x)\in C[-1, 1]$,

$$\Delta_h^j f(x) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} f(x+kh);\ -1\leq x\leq x+jh\leq 1,\ j=1, 2, \dots$$

and $\omega_j(f, t)=\sup\{|\Delta_h^j f(x)|: -1\leq x\leq 1-jt,\ 0\leq h\leq t\}$. Then we have the following

Theorem 1. *If $f(x)\in C[-1, 1]$ and there exist integers $0\leq i_1<i_2<\cdots<i_r$ such that*

$$\varepsilon_k=\operatorname{sgn} \Delta_h^{i_k} f(x);\ k=1, \dots, r, \tag{1}$$

*are independent of x and h when $-1\leq x<x+i_k h\leq 1$; $k=1, \dots, r$, then there exist polynomials $p_n(x)$ of degree at most n such that**

$$\|f(x)-p_n(x)\|\leq c(i_r)\,\omega(f, n^{-1}\log n) \tag{2}$$

* In what follows, $\|.\|$ will always denote supremum norm over the interval $[-1, 1]$.

and

(3) $$\operatorname{sgn} p_n^{(i_k)}(x)=\varepsilon_k;\ k=1,\ldots,r;\ |x|\le 1,$$

for all $n\ge 2i_r\cdot\log n$, *where the constant* $c(i_r)$ *depends only on* i_r.

Proof. The main idea is the use of iterated Stekloff-transforms as intermediate approximating means because they are well-approximating, sufficiently smooth, and their derivatives are strongly connected with the higher order differences of the function.

Let

(4) $$m=i_r[\log n/\log 2],\ h=4/n \text{ and } \lambda_n=1-mh,$$

and define the m times iterated Stekloff-transform of $f(x)$ in a slightly modified form as

$$f_{h,m}(x)=h^{-m}\int_0^h\ldots\int_0^h f(\lambda_n x+\sum_{k=1}^m t_k)\,dt_1\ldots dt_m;\quad |x|\le 1.$$

Then evidently

(5) $$|f(x)-f_{h,m}(x)|\le\|f(x)-f(\lambda_n x)\|+\|f(\lambda_n x)-f_{h,m}(x)|\le 2\omega(f,mh).$$

Moreover,

(6) $$f_{h,m}^{(j)}(x)=\lambda_n^j h^{-m}\int_0^h\ldots\int_0^h \Delta_h^j f(\lambda_n x-jh+\sum_{k=1}^{m-j} t_k)\,dt_1\ldots dt_{m-j};$$

$$j=0,\ 1,,\ldots,m;\quad |x|\le 1.$$

(This can be easily proved by induction on i.) Thus by (4) $\|f_{h,m}^{(m)}(x)| \le(\lambda_n/h)^m\|\Delta_h^m f(x)|\le h^{-m}\omega_m(f,h)\le 2^{-1}(2/h)^m\omega(f,h)=2^{m-1}n^m\omega(f,h) \le n^{m-i_r}\omega(f,h)$ and consequently $\omega(f_{h,m}^{(m-1)},1/n)\le n^{m-i_r-1}\omega(f,h)$.

Hence by a well-known theorem on simultaneous approximation of functions and derivatives (cf. e. g. P. O. Runck [2]), there exist polynomials $q_n(x)$ of degree at most n such that

(7) $$\|f_{h,m}^{(j)}(x)-q_n^{(j)}(x)|\le c_i n^{j-m+1}\omega(f_{h,m}^{(m-1)},1/n)\le c_i n^{j-i_r}\omega(f,h);$$

$$j=0,\ldots,m;\quad n>m,$$

where the c_i's are positive constants depending only on i. Therefore

(8) $$\|f_{h,m}^{(i_k)}(x)-q_m^{(i_k)}(x)\|\le\bar c(i_r)\omega(f,h);\ k=0,\ 1,\ldots,r,$$

$$\bar c(i_r)=\max\{c_{i_k}:1\le k\le r\}.$$

Using (6) and (1) we get that $\operatorname{sgn} f_{h,m}^{(i_k)}(x)=\varepsilon_k;\ k=0,1,\ldots,r;\ |x|\le 1$, i. e. by (8)

(9) $$\varepsilon_k q_n^{(i_k)}(x)\ge-\bar c(i_r)\omega(f,h);\quad k=0,\ 1,\ldots,r;\ |x|\le 1.$$

Thus the polynomial $q_n(x)$ "almost" satisfies the requirements (3), only a slight modification is needed. Let $d_r=1/i_r!$ and

(10) $$d_k=\frac{1}{i_k!}+\sum_{j=k+1}^{r} d_j\, 3^{i_j-i_k}\binom{i_j}{i_k};\quad k=r-1,\ r-2,\dots,1,$$

further $p_n(x)=q_n(x)+\bar{c}(i_r)\omega(f,h)\Sigma_{j=1}^{r} d_j\varepsilon_j(x+2)i_j$. By (5) and (7), this polynomial satisfies (2). Moreover, using (9) and (10) we get

$$\varepsilon_k p_n^{(i_k)}(x)=\varepsilon_k q_n^{(i_k)}(x)+\varepsilon_k\bar{c}(i_r)\omega(f,h)\sum_{j=k}^{r} d_j\varepsilon_j i_j(i_j-1)\dots(i_j-i_k+1)(x+2)^{i_j-i_k}$$

$$>\bar{c}(i_r)\omega(f,h)\Big[-1+d_k i_k!-\sum_{j=k+1}^{r} d_j 3^{i_j-i_k} i_j\dots(i_j-i_k+1)\Big]=0;\ |x|\leq 1,$$

i. e. (3) is also satisfied. q. e. d.

Comparing the estimate (2) to Jackson's order of convergence $O(\omega(f, 1/n))$, there is a loss $(\log n)$ in the accuracy. On the other hand, we can replace ω by ω_2, but then we cannot guarantee (3) on the whole interval $[-1, 1]$, as it can be seen from the following

Theorem 2. *If $f(x)\in C[-1, 1]$ and there exist integers $0\leq i_1<i_2<\dots<i_r$ such that* (1) *are independent of x and h when $-1\leq x<x+i_k h\leq 1$ $(k=1,\dots, r)$, then there exist polynomials $p_n(x)$ of degree at most n such that*

$$\|f(x)-p_n(x)\|\leq c^*(i_r)\omega_2(f, n^{-1}\log n)$$

and $\operatorname{sgn} p_n^{(i_k)}(x)=\varepsilon_k$, $k=1,\dots, r$; $|x|\leq 1-6i_r n^{-1}\log n$, *for all* $n>2i_r\log n$, *where $c^*(i_r)$ depends only on i_r.*

Let m and h be defined again by (4), and consider those linear functions $P(x)$ and $Q(x)$ for which

(11) $$P(-1+2mh)=f(-1+2imh),\ Q(1-2imh)=f(1-2imh);\ i=0, 1.$$

Then

(12) $$\max\{|f(x)-P(x)|: -1\leq x\leq -1+2mh\}\leq\omega_2(f, 2mh),$$

$$\max\{|f(x)-Q(x)|: 1-2mh\leq x\leq 1\}\leq\omega_2(f, mh).$$

Indeed, assume that the first maximum in (12) (denote it temporarily by M) is attained at $x_0\in(-1, 1+mh]$ (the proof is similar in case $x_0\in(-1+mh, -1+2mh)$). Then $2x_0+1\in(-1, 1+2mh)$ and using (11) we obtain

$$\omega_2(f, mh)\geq\omega_2(f, x_0+1)\geq|f(2x_0+1)-2f(x_0)+f(-1)|$$

$$=|f(2x_0+1)-P(2x_0+1)+2[f(x_0)-P(x_0)]+f(-1)-P(-1)|$$

$$\geq 2|f(x_0)-P(x_0)|-|f(2x_0+1)-P(2x_0+1)|$$

$$=2M-|f(2x_0+1)-P(2x_0+1)|\geq M.$$

The second relation in (12) can be similarly proved.

Now let

(13) $$\bar{f}(x)=\begin{cases} P(x) & \text{if } x\leq -1,\\ f(x) & \text{if } |x|\leq 1,\\ Q(x) & \text{if } x\geq 1,\end{cases}$$

then it is readily seen that $\omega_2(\bar{f}, mh) \leq 3\omega_2(f, mh)$. Define the m-times iterated Stekloff-transform of $\bar{f}(x)$ by

$$f_{h,m}(x) = (2h)^{-m} \int_0^h \ldots \int_0^h \sum_{\delta_k = \pm 1} \bar{f}\,(x + \sum_{k=1}^{m} \delta_k t_k) dt_1 \ldots dt_m,$$

where the outer summation is extended over all possible permutations of $\delta_1, \ldots, \delta_n$ with the values ± 1. Then evidently

$$\| f(x) - f_{h,m}(x) \| = \| \bar{f}(x) - f_{h,m}(x) \| \leq (1/2)\omega_2\,(\bar{f}, mh) \leq (3/2)\omega_2(f, mh).$$

Moreover

$$f_{h,m}^{(j)}(x) = (2h)^{-m} \int_{-h}^{h} \ldots \int_{-h}^{h} \Delta_{2h}^{j} \bar{f}(x - ih + \sum_{k=1}^{m-j} t_k) dt_1 \ldots dt_{m-j};$$

$$|x| \leq 1;\ j = 0, \ldots, m$$

(this can be proved by induction over i). Thus, (1) and (13) imply

$$\operatorname{sgn} f_{h,m}^{(i_k)}(x) = \varepsilon_k;\ k = 0, \ldots, r;\ |x| \leq 1 - mh.$$

The rest of the proof is the same as that of Theorem 1, henceforth we omit the details.

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