КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'77
София . 1980, с. 505—510

CONSTRUCTIVE FUNCTION THEORY'77
Sofia . 1980, p. 505—510

ON THE STRONG APPROXIMATION OF ORTHOGONAL SERIES

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Summary. Let $\{\varphi_n(x)\}$ be an orthogonal system on the interval (a, b). We consider the orthogonal series (1) $\sum_{n=0}^{\infty} c_n\varphi_n(x)$ with $\sum_{n=0}^{\infty} c_n^2<\infty$. It is well known that the series (1) converges in L^2 to a square-integrable function $f(x)$. Let us denote the n-th (C, α)-mean $(\alpha>-1)$ of the series (1) by $\sigma_n^{(\alpha)}(x)$.

Suppose that $0<\gamma<1$, $0<k<\gamma^{-1}$ and $1<\varkappa\leq 2$ furthermore there exists a number $p>1$ such that $p(p-1)^{-1}k\geq\varkappa$ and with this p for any $0<\delta<1$ and $2^m<n\leq 2^{m+1}$

$$\sum_{l=0}^{m}\{\sum_{v=2^l-1}^{\min(2^{l+1},n)} \alpha_{n\nu}^p(\nu+1)^{p(1-\delta)-1}\}^{1/p}=O(n^{-\delta}A_n),$$

where $(\alpha_{nk}/A_n)_{k,n=1}^{\infty}$ is a triangular matrix with $\alpha_{nk}\geq 0$ and $A_n=\sum_{k=0}^{n}\alpha_{nk}$. We give sufficient coefficient-conditions to assure the estimation

$$\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}\,|\sigma_\nu^{(\alpha)}(x)-f(x)|^k\}^{1/k}=o(n^{-\gamma})$$

almost everywhere in (a, b).

1. Let $\{\varphi_n\}$ be an orthogonal system on the interval (a, b). We consider the orthogonal series

$$\sum_{n=0}^{\infty} c_n\varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2<\infty. \tag{1}$$

It is well known that the series (1) converges in the space L^2 to a square-integrable function f. Let us denote the n-th partial sums of the series (1) by $s_n(x)$ and denote the n-th (C, α)-means of $\{s_n(x)\}$ and $\{nc_n\varphi_n(x)\}$ by $\sigma_n^{(\alpha)}(x)$ and $\tau_n^{(\alpha)}(x)$, respectively.

We consider a regular summation method T_n determined by a triangular matrix $(\alpha_{n\nu}/A_n)_{n,\nu=0}^{\infty}$, where $\alpha_{n\nu}\geq 0$ and $A_n=\sum_{\nu=0}^{n}\alpha_{n\nu}$.

Theorem 1. *Let* $0<\gamma<1$; $0<k<\gamma^{-1}$ *and* $1<\varkappa\leq 2$. *Suppose that there exist a number* $p>1$ *and a constant* K^* *such that*

* $K, K_1, K_2, \ldots$ will denote positive constants not necessarily the same at each occurrence.

(2) $$p'k \geqq \varkappa \quad (p' = p/(p-1))$$

and with this p for any $0<\delta<1$ *and* $2^m < n \leqq 2^{m+1}$

(3) $$\sum_{l=0}^{m} \{ \sum_{\nu=2^l-1}^{\min(2^{l+1},n)} \alpha_{n\nu}^p (\nu+1)^{p(1-\delta)-1} \}^{1/p} \leqq K n^{-\delta} A_n; \; n=1, 2, \dots$$

is fulfilled. Then using the notation

(4) $$L = L(\varkappa, p, k) = 1/2 + 1/\varkappa - 1/p'k$$

in the cases

(i) $L < \beta < \infty$

(5) $$\sum_{m=0}^{\infty} \{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 n^{2\gamma} \}^{\varkappa/2} < \infty,$$

(ii) $L = \beta$

(6) $$\sum_{m=0}^{\infty} \{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 n^{2\gamma} \log n \}^{\varkappa/2} < \infty,$$

and

(iii) $L - 1/2 < \beta < L$

(7) $$\sum_{m=0}^{\infty} \{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 n^{2(\gamma+L-\beta)} \}^{\varkappa/2} < \infty,$$

we have

(8) $$\{ \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} | \sigma_\nu^{(\beta-1)}(x) - f(x) |^k \}^{1/k} = o_x(n^{-\gamma})$$

almost everywhere in (a, b).

The case (i) is a generalization of a L. Leindler's result [3, Theorem 1].

We remark, without proof, that in the case of strong summability ($\gamma=0$) the following result is valid.

Theorem 2. *Let* $k>0$ *and* $1<\varkappa\leqq 2$. *If there exist a number* $p>1$ *and a constant K such that the condition* (2) *and*

(9) $$\{ \sum_{\nu=1}^{n} \alpha_{n\nu}^p \nu^{p-1} \}^{1/p} \leqq K \cdot A_n; \; n=1, 2, \dots$$

are satisfied and if the series (1) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* (a, b) *then using the notation* (4), *in the cases*

(i) $$L<\beta<\infty, \; \sum_{m=0}^{\infty} \{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \}^{\varkappa/2} < \infty,$$

(ii) $$L=\beta, \; \sum_{m=0}^{\infty} \{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \log n \}^{\varkappa/2} < \infty,$$

and

(iii) $$L-1/2<\beta<L,\ \sum_{m=0}^{\infty}\{\sum_{n=2^m+1}^{2^{m+1}} c_n^2 n^{2(L-\beta)}\}^{\varkappa/2}<\infty,$$

we have

$$A_n^{-1}\sum_{\nu=0}^{n}\alpha_{n\nu}\,|\sigma_\nu^{(\beta-1)}(x)-f(x)|^k=o_x(1)$$

almost everywhere in (a, b).

The case (i) is a generalization of a L. Leindler's result [2, Theorem 1] which itself contains a result of G. Sunouchi [4, Theorem 1], too.

Finally, we remark that in the cases $\alpha_{n\nu}=A_{n-\nu}^{(\alpha-1)}(\alpha>0;\ A_n^{(\alpha)}=(1+\alpha)(2+\alpha)\cdots(n+\alpha)/n!)$; $\alpha_{n\nu}=\nu^l$; $l\geqq-1$, $\alpha_{n\nu}=1/\nu\log\nu$ and so on, the conditions (3) and (9) are fulfilled.

2. We require the following lemmas.

Lemma 1. [1, p. 359]. *If* $r\geqq\varkappa>1$; $\gamma>0$; $\alpha>\gamma-1$ *and* $\beta\geqq\alpha+1/\varkappa-1/r$ *then*

$$\{\sum_{n=0}^{\infty}(n+1)^{r\gamma-1}|\tau_n^{(\beta)}(x)|^r\}^{1/r}\leqq K\{\sum_{n=0}^{\infty}(n+1)^{\varkappa\gamma-1}|\tau_n^{(\alpha)}(x)|^{\varkappa}\}^{1/\varkappa}.$$

Lemma 2 [6]. *Let* $0\leqq\gamma<1$ *and* $1\leqq\varkappa\leqq2$. *Then there exists a constant* K *such that*

$$\int_a^b(\sum_{n=1}^{\infty}n^{\varkappa\gamma-1}|\tau_n^{(\alpha)}(x)|^{\varkappa})dx\leqq\begin{cases}K\sum_{m=0}^{\infty}\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2n^{2\gamma}\}^{\varkappa/2} & \text{if } 1/2<\alpha<\infty,\\ K\sum_{m=0}^{\infty}\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2n^{2\gamma}\log n\}^{\varkappa/2} & \text{if } \alpha=1/2,\\ K\sum_{m=0}^{\infty}\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2n^{2\gamma+1-2\alpha}\}^{\varkappa/2} & \text{if } -1<\alpha<1/2.\end{cases}$$

We prove the following

Lemma 3. *Let* $0<\gamma<1$; $0<k<\gamma^{-1}$ *and* $1<\varkappa\leqq2$. *If there exist a number* $p>1$ *and a constant, such that the conditions* (2) *and* (3) *are satisfied then there exists a constant* K, *such that*

$$\int_a^b(\overline{\lim_{n\to\infty}}(\frac{n^{k\gamma}}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\tau_\nu^{(\beta)}(x)|^k)^{1/k})^{\varkappa}dx\leqq\begin{cases}K\sum_{m=0}^{\infty}\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2n^{2\gamma}\}^{\varkappa/2} & \text{if } L<\beta<\infty,\\ K\sum_{m=0}^{\infty}\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2n^{2\gamma}\log n\}^{\varkappa/2} & \text{if } L=\beta,\\ K\sum_{m=0}^{\infty}\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2n^{2(\gamma+L-\beta)}\}^{\varkappa/2} & \text{if } L-1/2<\beta<L.\end{cases}$$

Proof. Choosing $\delta=\gamma k$ and using the condition (3) by the Hölder's inequality we get for any $n=1, 2, \ldots$

(10) $$\sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(x)|^k \leqq K_1 n^{-\gamma k} A_n \{\sum_{\nu=0}^{n} (\nu+1)^{p'k\gamma-1} |\tau_\nu^{(\beta)}(x)|^{p'k}\}^{1/p'}.$$

Let now

(11) $$\alpha = \beta - 1/\varkappa + 1/p'k.$$

As $\beta > L - 1/2 (= 1/\varkappa - 1/p'k)$, that is $\alpha > 0$, and $0 < \gamma < 1$ we get $\alpha > \gamma - 1$, choosing $r = p'k$ and applying Lemma 1 we get

$$\{\sum_{\nu=0}^{\infty} (\nu+1)^{\gamma p'k-1} |\tau_\nu^{(\beta)}(x)|^{p'k}\}^{1/p'k} \leqq K_2 \{\sum_{\nu=0}^{\infty} (\nu+1)^{\varkappa\gamma-1} |\tau_\nu^{(\alpha)}(x)|^{\varkappa}\}^{1/\varkappa}.$$

Hence by (10) we have the following inequality

$$\int_a^b (\overline{\lim_{n\to\infty}} (\frac{n^{k\gamma}}{A_n} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(x)|^k)^{1/k})^{\varkappa} dx \leqq K_2 \int_a^b (\sum_{\nu=0}^{\infty} (\nu+1)^{k\gamma-1} |\tau_\nu^{(\alpha)}(x)|^{\varkappa}) dx.$$

Considering (11) by Lemma 2, we get the required result.

The method of proof of Lemma 3 has been given in G. Sunouchi [4].

Finally we consider the following obvious fact as a

Lemma 4. *Let $\{f_n(x)\}_1^\infty$ be a sequence of non-negative measurable functions defined in the interval (a, b). The sequence $\{f_n(x)\}$ converges to zero, almost everywhere in (a, b), if and only if, there exists a constant K such that for any* $\varepsilon > 0$ meas$\{x: \overline{\lim}_{n\to\infty} f_n(x) > \varepsilon\} \leqq K\varepsilon$.

3. Proof of Theorem 1. In this proof we combine the methods of G. Sunouchi [4] and L. Leindler [3]. First of all L. Leindler [3, p. 91] showed that (3) implies

(12) $$\sum_{\nu=0}^{n} a_{n\nu}(\nu+1)^{-\delta} \leqq K n^{-\delta} A_n;\ n = 1, 2, \ldots$$

for any $0 < \delta < 1$. Each of the conditions (5), (6) and (7) implies that $\sum_{n=1}^\infty c_n^2 n^{2\gamma} < \infty$ and G. Sunouchi [4] proved that under this condition

$$(1/A_n^{(\beta)}) \sum_{\nu=0}^{n} A_{n-\nu}^{(\beta-1)} |f(x) - s_\nu(x)| = o_x(n^{-\gamma});\ \beta > 0$$

almost everywhere in (a, b). Hence by the identity $\sigma_n^{(\beta)}(x) = (1/A_n^{(\beta)}) \sum_{\nu=0}^n A_{n-\nu}^{(\beta-1)} s_\nu(x)$ we get

(13) $$\sigma_n^{(\beta)}(x) - f(x) = o_x(n^{-\gamma});\ \beta > 0$$

almost everywhere in (a, b).

Using the identity $\tau_n^{(\beta)}(x) = -\beta(\sigma_n^{(\beta)}(x) - \sigma_n^{(\beta-1)}(x))$; $\beta > 0$ we obtain

$$\frac{1}{A_n} \sum_{\nu=0}^{n} a_{n\nu} |\sigma_\nu^{(\beta-1)}(x) - f(x)|^k$$
$$\leqq K_1 A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(x)|^k + K_2 A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\sigma_\nu^{(\beta)}(x) - f(x)|^k.$$

As $0 < \gamma k < 1$ by (12) and (13) we can see that $A_n^{-1} \sum_{\nu=0}^n a_{n\nu} |\sigma_\nu^{(\beta)}(x) - f(x)|^k = o_x(n^{-\gamma k})$; $\beta > 0$, sufficient to verify that

$$\lim_{n\to\infty} n^{\gamma k} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(x)|^k = 0 \tag{14}$$

almost everywhere in (a, b).

Let now

$$d_n = \begin{cases} c_n^2 n^{2\gamma}, \text{ if } L<\beta<\infty, \\ c_n^2 n^{2\gamma} \log n, \text{ if } L=\beta \\ c_n^2 n^{2(\gamma+L-\beta)}, \text{ if } L-1/2<\beta<L, \end{cases} \tag{15}$$

and let ε be a fixed arbitrary positive number. Then by (5), (6), (7) and (15) in the cases (i), (ii) and (iii) there exists a natural number $N=N(\varepsilon)$ such that

$$\sum_{m=N(\varepsilon)}^{\infty} \{ \sum_{n=2^m+1}^{2^{m+1}} d_n\}^{\varkappa/2} < \varepsilon^{\varkappa+1}. \tag{16}$$

We introduce the following notations

$$a_n = \begin{cases} c_n, \text{ if } n\leqq 2^{N(\varepsilon)}, \\ 0, \text{ if } n>2^{N(\varepsilon)}, \end{cases} \quad b_n = \begin{cases} 0, \text{ if } n\leqq 2^{N(\varepsilon)}, \\ c_n, \text{ if } n>2^{N(\varepsilon)}. \end{cases} \tag{17}$$

Denote $\sigma_\nu^{(\beta)}(a; x)$ and $\tau_\nu^{(\beta)}(a; x)$ the ν-th (C, β)-means of the series $\Sigma_{n=0}^{\infty} a_n \varphi_n(x)$ and the sequences $\{n a_n \varphi_n(x)\}$, respectively. Similarly, we use the symbols $\sigma_\nu^{(\beta)}(b; x)$ and $\tau_\nu^{(\beta)}(b; x)$ in connection with the series $\Sigma_{n=0}^{\infty} b_n \varphi_n(x)$. Easy to see that $\tau_\nu^{(\beta)}(x) = \tau_\nu^{(\beta)}(a; x) + \tau_\nu^{(\beta)}(b; x)$ and so

$$A_n^{-1} n^{\gamma k} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(x)|^k \leqq K_3 n^{\gamma k} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(a; x)|^k + K_3 n^{\gamma k} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(b; x)|^k. \tag{18}$$

First we consider the first term. We can choose a number ω such that $\gamma<\omega<1$ and $\omega<1/k$. Applying Lemma 3 with the series, $\Sigma a_n \varphi_n(x)$ by (15) and (17) we obtain

$$\int_a^b (\overline{\lim_{n\to\infty}} (n^{k\omega} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(a; x)|^k)^{1/k})^{\varkappa} dx \leqq K_4 \sum_{m=0}^{N(\varepsilon)-1} \{ \sum_{n=2^m+1}^{2^{m+1}} d_n\}^{\omega/2} < \infty$$

so $\overline{\lim}_{n\to\infty} n^{k\omega} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(a; x)|^k = O_x(1)$ almost everywhere in (a, b). It is clear that $\lim_{n\to\infty} n^{k(\gamma-\omega)} = 0$ and so

$$\lim_{n\to\infty} n^{\gamma k} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(a; x)|^k = 0 \tag{19}$$

almost everywhere in (a, b).

By (18) and (19) we have

$$\overline{\lim_{n\to\infty}}\, n^{\gamma k} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(x)|^k \leq \overline{\lim_{n\to\infty}}\, K_5 n^{\gamma k} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(b; x)|^k.$$

Using this inequality and applying Lemma 3 with the series $\Sigma b_n \varphi_n(x)$, by (15), (16) and (17) we obtain

$$\int_a^b (\overline{\lim_{n\to\infty}}\, (n^{\gamma k} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(x)|^k)^{1/k})^{\varkappa} dx \leq K_5 \sum_{m=N(\varepsilon)}^{\infty} \{ \sum_{n=2^m+1}^{2^{m+1}} d_n \}^{\varkappa/2} < K_5 \varepsilon^{\varkappa+1},$$

where the constant K_5 is independent of ε.

Hence

$$\operatorname{meas}\{x : \overline{\lim_{n\to\infty}}\, (n^{\gamma k} A_n^{-1} \sum_{\nu=0}^{n} a_{n\nu} |\tau_\nu^{(\beta)}(x)|^k)^{1/k} > \varepsilon\} < K_5 \varepsilon.$$

Finally applying Lemma 4 we have (14) almost everywhere in (a, b) so our proof is complete.

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Received September 14, 1977