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AN EXTENSION OF OKADA'S THEOREM FOR SUMMABILITY OF POWER SERIES

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Summary. Let $P(w)=\Sigma_{k=0}^{\infty} a_k w^k$ be a power series with positive radius of convergence, A_p any maximal domain into which P is uniquely analytically continuable and $\mathfrak{A}_p$ the set of all such A_p. Let $V=(c_{nk})$ be a matrix such that $C_n=\Sigma_{k=0}^{\infty} c_{nk}$, $n=0, 1, 2, \ldots$, exist. Denote by $\tau_n(w)=\Sigma_{k=0}^{\infty} c_{nk} \cdot \Sigma_{j=0}^{k} w^j$, $\sigma_n(w)=\Sigma_{k=0}^{\infty} c_{nk} \cdot \Sigma_{j=0}^{k} a_j w^j$. Consider a Banach sequence space B and denote by $Z_{V,B}$ the largest domain (not necessarily connected) such that for $w \in Z_{V,B}$ the sequence $(\tau_n(w))$ continuously belongs to B. If $0 \in Z_{V,B}$, then $(\sigma_n(w))$ continuously belongs to B for $w \in \mathfrak{G}_B=\bigcup_{A_p \in \mathfrak{A}_p} (A_p^c \cdot Z_{V,B}^c)^c$ (where $A \cdot B=\{a \cdot b \mid a \in A, b \in B\}$ and A^c=complement of A). A similar conclusion is valid if $\tau_n(w)$ replaced by $\Sigma_{k=0}^{\infty} c_{nk} w^k$ and $\sigma_n(w)$ by $\sigma_n(w)-C_n p(w)$. The results cover a recent theorem of Gawronski—Trautner (1976), extending a classical theorem of Y. Okada (1925) as well as new results, for instance on absolute summability, O- and o-theorems for rate of convergence and divergence, and others.

Introduction. Okada's theorem [9] describes a domain of compact summability for power series. Various extensions have been considered [5, 6, 10, 11]. B. L. R. Shawyer* posed the question whether Okada's theorem can also be formulated for absolute summability. The answer is affirmative as shown by Gawronski, Shawyer and Trautner. It is contained in a general Banach space version of Okada's theorem [4] which also covers other applications of interest, e. g. O- and o-theorems for rate of convergence, resp. divergence.

In this paper we will give a new proof of the results stated in [4]. There are two reasons for doing this. First, our proof is shorter and more direct than that of [4] which uses the deep-lying theorem of Aronszajn on separation of singularities [1]. Secondly, the new proof works under more general conditions than that of [4]. In addition, we will give some more examples of application.

The present paper is self-contained. Though we cannot avoid bringing again some arguments of [4], the set of repetitions will be of small measure.

* Personal communication March 1976.

1. Statement of Results. We will use the following notations. For sets $A, B, \ldots, \subseteq \mathbb{C}$ denote $A \cdot B = \{a \cdot b : a \in A,\ b \in B\}$, $a \cdot B = \{a \cdot b : b \in B\}$, A^c = complement of A, $A \odot B = (A^c \cdot B^c)^c$. For $A \subset \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ let $A^{-1} = \{a^{-1} : a \in A\}$.

Any power series

$$(1) \qquad P(w) = \sum_{k=0}^{\infty} a_k w^k$$

is assumed to have radius of convergence $R=1$; its partial sums are $s_n(w) = \sum_{k=0}^{n} a_k w^k$. Denote by A_P any maximal region (not necessarily simply connected) in $\mathbb{C}$ into which P can be continued uniquely analytically from $|w|<1$. Clearly A_P need not be determined uniquely; therefore denote by $\mathfrak{A}_P$ the set of all such A_P.

Let

$$(2) \qquad V = (c_{nk}); \quad n, k = 0, 1, 2, \ldots$$

be a summability matrix of sequence to sequence type, for which the row sums $c_n = \sum_{k=0}^{\infty} c_{nk}$; $n = 0, 1, 2, \ldots$ exist. Applying V to the sequence $\{s_n(w)\}_0^{\infty}$ we get the transformed sequence $\sigma_n(w) = \sum_{k=0}^{\infty} c_{nk} s_k(w)$; $n = 0, 1, 2, \ldots$ respectively the rest sequence $r_n(w) = \sigma_n(w) - c_n P(w)$; $n = 0, 1, 2, \ldots$ Taking particularly for P the geometric series we get (for $w \neq 1$)

$$\tau_n(w) = \sum_{k=0}^{\infty} c_{nk}\, (1 - w^{k+1})/(1-w) = c_n/(1-w) - (w/(1-w))\, v_n(w);$$

$$n = 0, 1, 2, \ldots,$$

where $v_n(w) = \sum_{k=0}^{\infty} c_{nk} w^k$.

Let B be a Banach space which is an algebraically linear subspace of the space Ω of all sequences $s = \{s_n\}_0^{\infty}$. So for any $s \in B$ the norm $\|s\|$ is defined. Let us write $v(w) = (v_n(w))_0^{\infty}$, $\tau(w) = \{\tau_n(w)\}_0^{\infty}$, $r(w) = \{r_n(w)\}_0^{\infty}$, $\sigma(w) = \{\sigma_n(w)\}_0^{\infty}$. These sequences are primarily considered as elements of the sequence space Ω for those $w \in \mathbb{C}$ for which they exist (i. e. the sums defining them are convergent). In addition, we will consider the subsets of $\mathbb{C}$ for which the above sequences are also elements of the Banach space B. All what will be done by the extended Okada theorem is the conclusion that if $\tau(w)$ continuously belongs to B for a certain set of $w \in \mathbb{C}$, then $\sigma(w)$ continuously belongs to B for some other set of $w \in \mathbb{C}$; a similar conclusion is valid from $v(w)$ to $r(w)$.

To give the precise formulation let us denote by $Z_{V,B}$ the largest domain (not necessarily connected) in $\mathbb{C}$, such that for $w \in Z_{V,B}$

a) the sums $\tau_n(w) = \sum_{k=0}^{\infty} c_{nk} \sum_{j=0}^{k} w^j$ exist for $n = 0, 1, 2, \ldots$

b) $\tau(w)$ belongs to B and is continuous (with respect to the norm $\|\cdot\|$).

Denote

$$(3) \qquad \mathfrak{G}^B = \bigcup \{A_P \odot Z_{V,B} : A_P \in \mathfrak{A}_P\}.$$

Theorem 1: *Given any power series P (1) and a summability matrix V (2) satisfying*

(4) $$0 \in Z_{V,B}.$$

Then $\sigma(w)$ belongs to B and is continuous for $w \in \mathfrak{G}_B$ (3).

Remark 1: The corresponding theorem in [4] still requires the conditions $\{c_n\}_0^\infty \in B$ and $1 \notin Z_{V,B}$ which are unnecessary.

Denote by $\widetilde{Z}_{V,B}$ the largest domain (not necessarily connected) in $\mathbb{C} \setminus \{1\}$ such that for $w \in \widetilde{Z}_{V,B}$

a) the sums $v_n(w) = \sum_{k=0}^{\infty} c_{nk} w^k$ exist for $n = 0, 1, 2, \ldots$

b) $v(w)$ belongs to B and is continuous.

Denote

(5) $$\widetilde{\mathfrak{G}}_B = \bigcup \{A_P \odot \widetilde{Z}_{V,B} : A_P \in \mathfrak{A}_P\}.$$

Theorem 2: *Given any power series P (1) and a summability matrix V (2) satisfying*

(6) $$0 \in \widetilde{Z}_{V,B}.$$

Then $r(w)$ belongs to B and is continuous for $w \in \widetilde{\mathfrak{G}}_B$ (5).

Remark 2: By definition $1 \notin \widetilde{Z}_{V,B}$ which implies

(7) $$\widetilde{\mathfrak{G}}_B \subseteq \bigcup \{A_P : A_P \in \mathfrak{A}_P\}.$$

This restriction is quite natural since $r(w)$ usually can be defined at most for such w for which P is analytic. In many cases there will hold $B \subseteq c_0 = \{\{s_n\}_0^\infty : s_n \to 0\}$ and $c_n \to 1$; the $1 \notin \widetilde{Z}_{V,B}$ will be fulfilled automatically.

Remark 3: One may consider also $\widehat{r}_n(w) = \sigma_n(w) - P(w) = r_n(w) - P(w)(1 - c_n)$. Then $\widehat{r}(w)$ belongs to B and is continuous for $w \in \widetilde{\mathfrak{G}}_B$ under (6) and the additional condition

(8) $$\{1 - c_n\}_0^\infty \in B.$$

2. Proofs. We first give the proof of theorem 2 which is the more complicated one due to $1 \notin \widetilde{Z}_{V,B}$ and (7); theorem 1 then will be obtained by a slight modification of arguments.

Proof of theorem 2. 1st step: Let Γ be a rectifiable curve, χ a complex-valued function of bounded variation defined on Γ, f a continuous map from Γ to B, then $\int_\Gamma f(t)d\chi(t)$ defined as Riemann—Stieltjes-integral, belongs to B and satisfies $|\int_\Gamma f(t)d\chi(t)| \leq \sup\{|f(t)| \int_\Gamma |d\chi(t)| : t \in \Gamma\}$ (see e. g. [2]). More generally let f be continuous on the set $A \cdot \Gamma$, where A is compact, then $g(w) = \int_\Gamma f(wt)\, d\chi(t)$ belongs to B and is uniformly continuous for $w \in A$; the latter statement follows from $\|g(w_2) - g(w_1)\| \leq \int_\Gamma \|f(w_2 t) - f(w_1 t)\| \cdot |d\chi(t)|$ and the uniform continuity of f on the compact set $A \cdot \Gamma$.

2nd step: By (6) the power series $v_n(w)=\Sigma_{k=0}^{\infty} c_{nk} w^k$ are uniformly convergent for $|w|\leq r_0$ and some $0<r_0<1$. Expressing the coefficients a_n by a Cauchy-integral, one gets

$$s_k(w)=(1/2\pi i)\int_{|t|=1/2} P(t)\,\frac{1-(w/t)^{k+1}}{1-w/t}\,\frac{dt}{t}$$

and for $|w|\leq r_0/2$

$$\sigma_n(w)=(1/2\pi i)\int_{|t|=1/2} P(t)\tau_n(w/t)\,\frac{dt}{t}=(C_n/2\pi i)\int_{|t|=1/2}\frac{P(t)}{t-w}\,dt \tag{9}$$

$$-(w/2\pi i)\int_{|t|=1/2}\frac{v_n(w/t)}{t-w}\,\frac{P(t)}{t}\,dt$$

which implies

$$r(w)=-(w/2\pi i)\int_{|t|=1/2}\frac{v(w/t)}{t-w}\,\frac{P(t)}{t}\,dt. \tag{10}$$

3rd step: From (10) we may immediately conclude that $r(w)\in B$ continuously for $|w|\leq r_0/2$. We have to show that this holds for all $w\in\widetilde{\mathfrak{G}}_B$ (5).

Given any $w_0\in\widetilde{\mathfrak{G}}_B$, then for some $A_P\in\mathfrak{A}_P$ we have $w_0\in A_P\odot\widetilde{Z}_{V,B}$. This may be formulated equivalently as $w_0\cdot A'\subset\widetilde{Z}_{V,B}$, where $A'=(A_P^c)^{-1}$ [5, Lemma 2]. Here A_P^c is considered as subset of $\overline{\mathbb{C}}$, so $\infty\in A_P^c$ and $0\in A'$. Since A' is compact and $\widetilde{Z}_{V,B}$ is open, we can find an ε-neighbourhood $U\varepsilon(A')$ such that $w_0\cdot U\varepsilon(A')\subset\widetilde{Z}_{V,B}$. The bounded open set $U\varepsilon(A')$ has only a finite number of disjoint components of connectivity, say $M_0, M_1,\ldots, M_q$, where $0\in M_0$, say. We now can find Jordan curves $\gamma_0, \gamma_1,\ldots,\gamma_q$ such that

i) $\gamma_j\subset M_j\setminus A'$; $j=0,1,2,\ldots,q$

ii) $A'\subset\bigcup_{j=0}^{q} B_j$, where B_j denotes the region interior of γ_j

iii) $0\in B_0$.

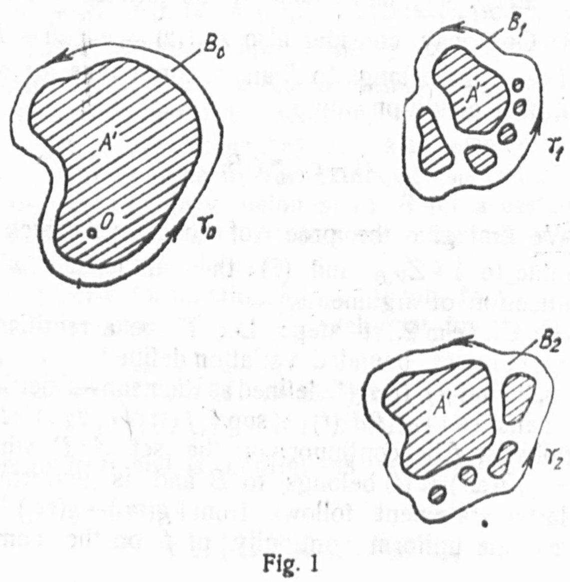

Fig. 1

Define for $j=0, 1, \ldots, q$ Jordan curves $\Gamma_j=\gamma_j^{-1}$ with positive orientation. From i) we get $\Gamma_j \subset A_P$, from ii) we get $A_P^c \subset \bigcup_{j=0}^{q} B_j^{-1}$, where B_j^{-1} is the interior region of Γ_j for $j=1, 2, \ldots, q$, but B_0^{-1} is the exterior region of Γ_0 by iii). From $w_0 \cdot U_\varepsilon(A') \subset \widetilde{Z}_{V,B}$ we conclude $w_0 \cdot \bigcup_{j=0}^{q} \overline{B}_j \subset \widetilde{Z}_{V,B}$, so $1 \notin w_0 \cdot \bigcup_{j=0}^{q} \overline{B}_j$ (since $1 \notin \widetilde{Z}_{V,B}$ by definition). This means $w_0^{-1} \notin \bigcup_{j=0}^{q} \overline{B}_j$ or $w_0 \notin \bigcup_{j=0}^{q} \overline{B}_j^{-1}$.

Consider for $n=0, 1, 2, \ldots$ the functions

$$g_n(w)=\Big(\int_{\Gamma_0}-\int_{\Gamma_1}-\cdots-\int_{\Gamma_q}\Big)\frac{w}{2\pi i}\,\frac{v_n(w/t)}{t-w}\,\frac{P(t)}{t}\,dt.$$

Clearly $g_n(w)$ is analytic in the domain $D=(\bigcup_{j=0}^{q} \overline{B}_j^{-1})^c \cap \{|w|<\widetilde{R}_0 \cdot T\}$, where $R_0=\sup\{|w|: w \in \widetilde{Z}_{V,B}\}$, $T=\inf\{|t|: t \in \bigcup_{j=0}^{q} \Gamma_j\}>1$.

We claim that $g_n(w)=-r_n(w)$ in D. This is obvious for a small neighbourhood of 0 by (10) and Cauchy's theorem. In general, this follows by approximating the power series defining r_n and v_n by their partial sums uniformly for $|w|<\widetilde{R}_0 \cdot T-\varepsilon$. So we get for $w \in D$

$$r(w)=\Big(-\int_{\Gamma_0}+\int_{\Gamma_1}+\int_{\Gamma_q}\Big)\frac{w}{2\pi i}\,\frac{v(w/t)}{t-w}\,\frac{P(t)}{t}\,dt.$$

For $t \in \Gamma_j$ we have $w_0/t \in w_0 \cdot \gamma_j \subset w_0 \cdot M_j \subset w_0 \cdot U_\varepsilon(A') \subset \widetilde{Z}_{V,B}$, so $v(w_0/t)$ is continuous for $t \in \Gamma_j$. The same statement holds in some neighbourhood $U_\varepsilon(w_0)$. So by our 1st step we get $r(w) \in B$ and $r(w)$ is continuous in $U_\varepsilon(w_0)$.

Proof of theorem 1: We follow the proof of theorem 2 and have to do only small changes. From (4) we get (9). Given any $w_0 \in \mathfrak{G}_P$ (3) then for some $A_P \in \mathfrak{A}_P$ we have (with $A'=(A_P^c)^{-1}$) $w_0 \cdot A' \subset Z_{V,B}$. As above, we consider sets $M_0, M_1, \ldots, M_q$, Jordan curves $\gamma_0, \gamma_1, \ldots, \gamma_q$ with i), ii) and iii). For the inverse Jordan curves $\Gamma_j=\gamma_j^{-1}$ with positive orientation we find $\Gamma_j \subset A_P$, $A_P^c \subset \bigcup_{j=0}^{q} B_j^{-1}$, where B_j^{-1} is the interior region of Γ_j for $j=1, \ldots, q$, but B_0^{-1} is the exterior region of Γ_0.

So by Chauchy's theorem we may replace the path of integration in (9) by $\Gamma_0, \Gamma_1, \ldots, \Gamma_q$ and we get for $|w|<R_0 \cdot T$, $R_0=\sup\{|w|: w \in Z_{V,B}\}$

$$\sigma(w)=\Big(\int_{\Gamma_0}-\int_{\Gamma_1}-\cdots-\int_{\Gamma_q}\Big)\frac{\tau(w/t)}{2\pi i}\,\frac{P(t)}{t}\,dt.$$

Now we conclude that $\sigma(w) \in B$ continuously in some neighbourhood of w_0.

3. Applications. We list up various applications of the above theorems obtained by different choices of the Banach space B. Before, we remark that continuity of any sequence $s(w)=\{s_n(w)\}_0^\infty$ in an open set G is equivalent to uniform continuity on any compact subset K of G.

Applications of theorem 1. 1. Take $B=c=\{s=\{s_n\}_0^\infty : \lim_{n\to\infty} s_n$ exists, $\|s\|=\sup_n |s_n|\}$ the space of convergent sequences. Then $Z_{V,c}$ is the maximal open set such that $\tau(w)=(\tau_n(w))$ is uniformly continuous on any compact subset K, which means uniform convergence on K. Theorem 1 then implies uniform convergence of $\sigma(w)=(\sigma_n(w))$ on any compact subset K of $\mathfrak{G}_c=\bigcup_{A_P\in\mathfrak{A}_P} A_P \odot Z_{V,c}$. If in addition the limit of $(\tau_n(w))$ is $(1-w)^{-1}$ then the limit of $\sigma_n(w)$ is $P(w)$. This yields the classical Okada theorem in the version of W. Gawronski and R. Trautner [5]. In general $\tau_n(w))$ may converge to different analytic function in different components of $Z_{V,c}$, (W. Luh [8]), in which case $\lim_{n\to\infty}\sigma_n(w)$ may be different from the analytic continuation of P.

2. Consider a weight sequence $w=(w_n)$, $w_n>0$, and take $B=m_w=\{s=\{s_n\}_0^\infty : \|s\|=\sup_n |s_n/w_n|<\infty\}$. Then Okada's theorem yields a domain $\mathfrak{G}m_w$, such that $\sigma_n(w)=O(w_n)$ on each compact subset. If $w_n\nearrow\infty$ we get an O-theorem for rate of divergence. Of particular interest is the case $w_n=a^n$, $a>1$.

3. Take for the above weight sequence $w=(w_n)$ the space

$$B=m_{w,0}=\{s=\{s_n\}_0^\infty : \lim_{n\to\infty}(s_n/w_n)=0,\ \|s\|=\sup_n |s_n/w_n|\}.$$

We get a similar o-theorem for rate of divergence. In many cases there will hold $Z_{V,w}=Z_{V,w,0}$ and consequently $\mathfrak{G}_w=\mathfrak{G}_{w,0}$.

4. Take for the above weight sequence $w=(w_n)$ the space

$$B=l^p_w=\{s=\{s_n\}_0^\infty : \|s\|=(\sum_{n=0}^{\infty}|s_n|^p/w_n)^{1/p}<\infty\},\qquad 1\leq p<\infty.$$

We then get Okada's theorem for l^p_w-spaces. The above space m_w may also be interpreted as l^∞_w.

Applications of theorem 2: 1. Take $B=c_0=\{s=\{s_n\}_0^\infty : \lim_{n\to\infty} s_n=0$, $\|s\|=\sup_n |s_n|\}$ the space of zero sequences. If (8) is satisfied we get again the result of [5].

2. Take $B=m=\{s=\{s_n\}_0^\infty$, $\|s\|=\sup_n |s_n|<\infty\}$, then we get Okada's theorem for bounded sequence. Clearly $Z_{V,c_0}\subseteq Z_{V,m}$; if $(c_n-1)\in c_0$ and $Z_{V,c_0}\neq\emptyset$, then even $Z_{V,c_0}=Z_{V,m}$.

3. Consider a weight sequence $w=\{w_n\}_0^\infty$, $w_n>0$, $\lim_{n\to\infty} w_n=0$ and take $B=m_w=\{s=\{s_n\}_0^\infty : \|s\|=\sup_n |s_n/w_n|<\infty\}$. Then $\mathfrak{G}_{m_w}$ is a domain, such that $r_n=0(w_n)$ on each compact subset, i. e. we get an O-theorem for rate of convergence.

4. Take for the above weight sequence the space $B=m_{w,0}=\{s=\{s_n\}_0^\infty : \lim_{n\to\infty}(s_n/w_n)=0,\ \|s\|=\sup_n |s_n/w_n|\}$. We get a similar o-theorem for rate of convergence.

Okada's theorem for rate of convergence may be applied to analytic continuation problems. See for instance D. Gaier [3] who used it to prove Fabry's gap theorem, and W. Künneth [7].

5. Take for the above weight sequence $w=(w_n)$ the space $B=l^p_w=\{s=[s_n]_0^\infty : \|s\|=(\sum_{n=0}^{\infty}|s_n|^p/w_n)^{1/p}<\infty\}$ we get Okada's theorem for l^p_w-spaces.

6. Take $B=A_p=s=\{s_n\}_0^\infty$, $\|s\|=(|s_1|^p+\sum_{n=0}^{\infty}|\Delta s_n|^p)^{1/p}<\infty\}$. This yields p-absolute summability, for $p=1$ ordinary absolute summability.

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