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THEOREMS OF LITTLEWOOD-PALEY TYPE FOR BMO AND FOR ANISOTROPIC HARDY SPACES

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Summary. Some special properties are proved for the space BMO of functions of bounded mean oscillation, the isotropic Hardy spaces in the sense of Fefferman-Stein, and their anisotropic generalizations in the sense of Calderón-Torchinsky (Theorems of Littlewood-Paley type).

1. Introduction. Let R_n be the Euclidean n-space. It is well known that theorems of Littlewood-Paley type for the classical Lebesgue spaces L_p on R_n with $1<p<\infty$ play an important role in the theory of function spaces. So it is natural to ask for corresponding theorems for other function spaces defined on R_n. For the Hardy spaces H_p on R_n with $1\geq p>0$ (in the sense of C. Fefferman and E. M. Stein, [3]) such a theorem has been given by J. Peetre [7, 8]. On the other hand, A. P. Calderón and A. Torchinsky generalized in [1, II] the idea of the Hardy spaces on R_n, they introduced anisotropic Hardy spaces (and also more general Hardy spaces). The main aim of this paper is the proof of theorems of Littlewood-Paley type for the anisotropic Hardy spaces in the sense of A. P. Calderón and A. Torchinsky (Theorem 1). In the isotropic case, using Fefferman's famous result that the dual of H_1 is BMO, the space of functions of bounded mean oscillation on R_n, we shall prove a corresponding result for BMO (Theorem 2).

Section 2 contains the needed definitions and some preliminaries. In Section 3, we prove the two theorems.

2. Definition and Preliminaries. 2.1. The Spaces F_p. R_n is the real Euclidean n-space. S is the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions on R_n, and S' is the dual of S (the space of all tempered distributions on R_n). The Fourier transform and the inverse Fourier transform on S' are denoted by F and F^{-1}, respectively.

Let Z be the set of all $f\in S$ such that supp Ff is compact and $(Ff)(x)=0$ near the origin. If $FZ=D(R_n-\{0\})=C_0^\infty(R_n-\{0\})$ is equipped in the usual way with a locally convex topology, then this topology can be carried over via F (or F^{-1}) to Z. If Z' is the corresponding dual space, then the Fourier transform Ff [inverse Fourier transform $F^{-1}f$] of $f\in Z'$ is defined by

$$(Ff)(\varphi)=f(F\varphi), \quad [(F^{-1}f)(\varphi)=f(F^{-1}\varphi)],$$

for all $\varphi\in FZ=F^{-1}Z=D(R_n-\{0\})$. In other words, F and F^{-1} are one-to-one-mappings from Z' onto $D'(R_n-\{0\})$.

If $a_1, \dots, a_n$ are n-positive numbers (in the sequel we assume, without loss of generality, $a_j\geqq 1$), then

$$d(x)=\sum_{j=1}^{n} |x_j|^{1/a_j} \tag{1}$$

is an anisotropic distance-function. Furthermore, we consider the anisotropic dyadic covering of $R_n-\{0\}=\bigcup_{k=-\infty}^{\infty} Q_k$,

$$Q_k=\{x: |x_j|\leqq 2^{(k+1)a_j};\ j=1,\dots,n\}-\{x: |x_j|\leqq 2^{(k-1)a_j},\ j=1,\dots,n\}.$$

Let Φ be the set of all systems $\varphi=\{\varphi_k(x)\}_{k=-\infty}^{\infty}\subset S$ of functions, such that for any multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$

$$\sup\{[d(x)]^{\alpha_1 a_1+\dots+\alpha_n a_n}|D^\alpha\varphi_k(x)|: x\in R_n,\ k=0,\pm1,\pm2,\dots\} \tag{2}$$
$$=M<\infty, \quad \operatorname{supp}\varphi_k\subset Q_k.$$

Φ_1 consists of all systems $\varphi\in\Phi$ with

$$\sum_{k=-\infty}^{\infty}\varphi_k(x)=1 \quad \text{if} \quad x\in R_n-\{0\}.$$

It is not hard to see that Φ_1 is not empty. $L_p=L_p(R_n)$ with $0<p\leqq\infty$ and $\|\cdot\|_{L_p}$ have the usual meaning (Lebesgue measure on R_n).

Definition 1. *If* $0<p<\infty$, *then*

$$F_p=\{f\,|\,f\in Z',\ \|f\|_{p,\varphi}=\|(\sum_{k=-\infty}^{\infty}|(F^{-1}\varphi_k Ff)(x)|^2)^{1/2}\|_{L_p}<\infty \ \text{for all}\ \varphi\in\Phi\}. \tag{3}$$

Remark 1. The definition is meaningful, because, by the Paley-Wiener-Schwartz theorem, $F^{-1}\varphi_k Ff$ is an analytic function. F_p coincides with the (homogeneous anisotropic) spaces $\dot{F}^0_{p,2}$ introduced in [10, 2.5.2, p. 105]. (For technical reasons, in [10], the corridors Q_k are subdivided into rectangles. But this is immaterial.) In particular, F_p is a quasi-normed (normed if $1\leqq p<\infty$) complete space, where all the quasi-norms $\|f\|_{p,\varphi}$, $\varphi\in\Phi_1$, are equivalent to each other. F_p is considered as a subspace of Z'. The spaces F_p depend on $a_1,\dots,a_n$, nevertheless we shall not indicate this dependence ($a_1,\dots,a_n$ are fixed once and for all, if it is not stated otherwise). One can show that a corresponding space F_∞ is not reasonable (in particular, the quasi-norms $\|f\|_{\infty,\varphi}$, $\varphi\in\Phi_1$, are not equivalent to each other).

2.2. The Spaces $\widetilde{F}_p$. All symbols have the same meaning as in the previous subsection.

Definition 2. *If* $1<p\leqq\infty$, *then*

(4) $$\widetilde{F}_p=\{f:f\in Z',\ \exists\{\varphi_k\}_{k=-\infty}^{\infty}\in\Phi_1,\ \ \exists\{f_k\}_{k=-\infty}^{\infty}\subset L_p \text{ with } \|(\sum_{k=-\infty}^{\infty}|f_k(x)|^2)^{1/2}\|_{L_p}<\infty \text{ and } f=\sum_{Z'\ k=-\infty}^{\infty}F^{-1}\varphi_k Ff_k\}.$$

Remark 2. $\widetilde{F}_p$ can be normed by $\|f\|_{\widetilde{F}_p}=\inf\|(\Sigma_{k=-\infty}^{\infty}|f_k(x)|^2)^{1/2}\|_{L_p}$, where the infimum is taken over all admissible representations of f in the sense of (4). Obviously, $F_p\subset\widetilde{F}_p$ if $1<p<\infty$. Using the Hilbert space version of the anisotropic Michlin-Hörmander multiplier theorem for L_p with $1<p<\infty$, then it is not hard to see that $F_p=\widetilde{F}_p$ if $1<p<\infty$.

Hence, only the space $\widetilde{F}_\infty$ is of interest. It is possible to show that the definition of a corresponding space $\widetilde{F}_1$ is not very reasonable. ($\widetilde{F}_p$ with $0<p<1$ is meaningless, because Fg with $g\in L_p$ is not defined, if $0<p<1$).

2.3. The Spaces H_p. Again, all the symbols have the same meaning as above. A_t,

$$A_t x=(t^{a_1}x_1,\ldots,t^{a_n}x_n),\quad t>0,\quad x=(x_1,\ldots,x_n)\in R_n,$$

is a continuous group of affine transformations of R_n in the sense of A. P. Calderón and A. Torchinsky, [1, I, p. 2]. (The groups in [1, I] are defined by qualitative terms. But the above groups are an interesting and perhaps the most important example.) $d(x)$, defined by (1), coincides essentially with the distance-function $\varrho(x)$ in [1, I], p. 6 (there exist two positive numbers c_1 and c_2 such that $c_1d(x)\leqq\varrho(x)\leqq c_2d(x)$ for every $x\in R_n$). If $\lambda\in S$, with $\lambda(x)=1$ near the origin, has a compact support, and if $b>0$, then

(6) $$M_b(x,f)=\sup\{|(F^{-1}\lambda(A_t\,.)Ff)(x-y)|:d(y)\leqq bt\},\ f\in S',$$

is the anisotropic non-tangential maximal function, which coincides essentially with corresponding function in [1, I, p. 8]. (If $a_1=\ldots=a_n=a\geqq1$, isotropic case, then $M_b(x,f)$ is essentially the same as $u^*(x)$ in [3], p. 183.)

Definition 3. *If* $0<p<\infty$, *then*

(7) $$H_p=\{f:f\in S',\ \ M_b(x,f)\in L_p\}.$$

Remark 3. These are the anisotropic Hardy spaces, introduced by A. P. Calderón and A. Torchinsky [1, II], Definition 1.1. We recall that L_p are the usual Lebesgue-spaces on R_n. $\|M_b(\cdot,f)\|_{L_p}$ is a quasi-norm on H_p. The space H_p does not depend on λ and b (equivalent quasi-norms), but it depends on $a_1,\ldots,a_n$ (we recall that $a_1,\ldots,a_n$ are fixed and that we do not indicate this dependence). If $a_1=\ldots=a_n\geqq1$, then one obtains the usual (isotropic) Hardy spaces, cf. [3].

2.4. The Space BMO. Definition 4. *BMO, the space of functions of bounded mean oscillation on* R_n, *is the set of all complex-valued locally integrable functions on* R_n *such that*

$$\|f\|_{BMO}=\sup_Q\inf_{c_Q}(1/|Q|)\int_Q|f(x)-c_Q|\,dx<\infty,$$

where Q *stands for cubes in* R_n, *and* c_Q *are arbitrary complex numbers*.

Remark 4. This is the famous Banach space BMO introduced by F. John and L. Nirenberg in [4]. It holds

$$\inf_{C} \int_{R_n} \frac{|f(x)-C|}{1+|x|^{n+1}}\, dx \leqq c\, \| f \|_{\mathrm{BMO}}. \tag{8}$$

A proof of this assertion may be found in [3, p. 141/142]. For further infomations concerning BMO we refer to [9].

2.5. Convention. All spaces will be considered as subspaces of Z'. For F_p and $\widetilde{F}_p$ this is obvious by definition. Furthermore, if $f \in S'$, then the restriction of $f(\varphi)$ to $\varphi \in Z$, yields an element of Z', denoted by $f_{Z'}$. Assume that $f_{Z'}=g_{Z'}$ for $f \in S'$ and $g \in S'$. Then it is not hard to see that $f=g+P$, where P is a polynomial. In other words, we have $S' \subset Z'$, provided that the elements of S' are considered modulo polynomials. Let $f \in H_p$ and $f+P \in H_p$, where P is a polynomial. Then $P \in H_p$. Now, (6) yields $L_p \ni M_b(x,P) \geqq |P(x)|$. Hence, $P(x)\equiv 0$. This shows that it is meaningful to consider H_p as a subspace of Z'. Next we assume that $f \in \mathrm{BMO}$ and $f+P \in \mathrm{BMO}$. Again, we have $P \in \mathrm{BMO}$. Now, (8) yields $P(x)\equiv \mathrm{const}$. Obviously, $\|f\|_{\mathrm{BMO}} = \|f+\mathrm{const}.\|_{\mathrm{BMO}}$. Hence, it is meaningful to consider BMO as a subspace of Z'. Finally we add the following remark. Z is dense in H_p, cf. [1, II], Theorem 1.8 (v), and also dense in F_p, cf. [10, p. 88]. Then, with the usual procedure, the dual spaces $(H_p)'$ and $(F_p)'$ can be interpreted as subspaces of Z'. All dual spaces in this paper must be understood in this way.

3. Littlewood-Paley Theorems. 3.1. Littlewood-Paley Theorem for H_p. All symbols have the above meaning, in particular $a_j \geqq 1$ are given numbers.

Theorem 1. *If* $0<p<\infty$, *then*

$$F_p = H_p. \tag{9}$$

Proof. Step 1. If $1<p<\infty$, then

$$H_p = F_p = L_p, \tag{10}$$

where L_p are the usual Lebesgue spaces on R_n. $H_p = L_p$ (if $1<p<\infty$) has been proved in [1, II], Theorem 1.2, and $F_p = L_p$ is a well-known classical (anisotropic) Littlewood-Paley theorem for the L_p-spaces (it is an easy consequence of the Hilbert space version of anisotropic multiplier theorems of Michlin-Hörmander type for L_p, cf. [5, p. 69/70] and [6, p. 211]). So we may restrict our attention to $0<p\leqq 1$, in the sequel. Furthermore, Z is dense in H_p (cf. [1, II, Theorem 1.8 (v)]) and dense in F_p (cf. [10, p. 88]). Hence, it will be sufficient to prove the equivalence of the quasi-norms of H_p and F_p for $f \in Z$.

Step 2. We recall that $m(x) \in L_\infty$ is said to be a (Fourier) multiplier for H_p if $F^{-1}[mFf]$ (first defined on Z, afterwards extended by continuity) is a bounded mapping from H_p into itself. A. P. Calderón and A. Torchinsky proved in [1 II], Theorem 4.6, a multiplier criterion for the spaces H_p with $0<p\leqq 1$, which is the counterpart of the well-known Hörmander criterion for the spaces L_p with $1<p<\infty$. Using this result, it follows that any function $m(x) \in L_\infty$ is a multiplier for H_p if

$$\sup_{t>0} \|(1+d(x))^{\varkappa}[Fm(t^{a_1}\cdot, \ldots, t^{a_n}\cdot)h(\cdot)](x)\|_{L_2(R_n)} = M < \infty.$$

Here $d(x)$ has been defined in (1), $h(x)\in S$ is an appropriate fixed function with compact support and $h(x)=0$ near the origin, $\varkappa>(\Sigma_{j=1}^{n}a_j)(1/p-1/2)$, and $0<p\leq 1$. It follows easily that any function $m(x)$ with

$$\sup_{t>0}\sum_{|\alpha|\leq L}\|D_x^\alpha m(t^{a_1},\ldots,t^{a_n}\cdot)\|_{L_2(Q)}=M<\infty$$

is a multiplier for H_p provided that L is sufficiently large, and $Q=\{x\mid\varepsilon<|x|<\varepsilon^{-1}\}$ is an appropriate set. This proves that any function $m(x)$ with

$$\sup\{[d(x)]^{\alpha_1 a_1+\ldots+\alpha_n a_n}|D^\alpha m(x)|: x\in R_n-\{0\}, |\alpha|\leq L\}=M<\infty, \tag{11}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$, is a multiplier for H_p, provided that L (depending on p) is large enough.

Step 3. We consider the Hilbert space version of H_p. If $f=\{f_j\}_{j=-\infty}^{\infty}$ with $f_j\in S'$, then $M_b(x,f)$ and $H_p(l_2)$ are defined by (6) and (7), respectively, where $|\cdot|$ in (6) must be replaced by $\|\cdot\|_{l_2}$ (and $f\in S'$ in (7) by $\{f_j\}\subset S'$). We take it for granted (without checking it in detail) that the multiplier theorem for H_p, formulated in (11), is also valid (after obvious modifications) for $H_p(l_2)$. (For a corresponding assertion concerning the usual Hardy spaces we refer to [3], p. 167.) More precisely: $m(x)=\{m_{j,k}(x)\}_{j,k=-\infty}^{\infty}\in L_\infty(l_2\to l_2)$ is a multiplier for $H_p(l_2)$ if

$$\sup\{[d(x)]^{\alpha_1 a_1+\ldots+\alpha_n a_n}\Big(\sum_{j,k=-\infty}^{\infty}|D^\alpha m_{j,k}(x)|^2\Big)^{1/2}: x\in R_n-\{0\},\ |\alpha|\leq L\}=M<\infty, \tag{12}$$

provided that L is large enough. If $\varphi=\{\varphi_k\}_{k=-\infty}^{\infty}\in\Phi$, and $m_{k,0}=\varphi_k$ and $m_{j,k}=0$ otherwise, then (2) shows that (12) is satisfied. Applying the just mentioned multiplier theorem for $H_p(l_2)$ to the vector $(\ldots,0,f,0,\ldots)$, where $f\in Z$ occupies the place "0", then one obtains that

$$\|\{F^{-1}\varphi_j Ff\}\|_{H_p(l_2)}\leq c\|f\|_{H_p}, \tag{13}$$

where c is independent of $f\in Z$. Here $\|f\|_{H_p}=\|M_1(\cdot,f)\|_{L_p}$, similarly for $H_p(l_2)$. Let $\psi=\{\psi_k\}_{k=-\infty}^{\infty}\in\Phi$, and $\tilde{m}_{0,k}=\psi_k$ and $\tilde{m}_{j,k}=0$ otherwise. If one applies the multiplier theorem for $H_p(l_2)$ with $\tilde{m}_{j,k}$ instead of $m_{j,k}$ to $\{F^{-1}\varphi_j Ff\}_{j=-\infty}^{\infty}\in H_p(l_2)$, where $f\in Z$, then

$$\Big\|\sum_{j=-\infty}^{\infty}F^{-1}\varphi_j\psi_j Ff\Big\|_{H_p}\leq c\|\{F^{-1}\varphi_j Ff\}\|_{H_p(l_2)}. \tag{14}$$

With an appropriate choice of $\psi\in\Phi$, (13) and (14) yield

$$\|f\|_{H_p}\sim\|\{F^{-1}\varphi_j Ff\}\|_{H_p(l_2)},\quad \varphi\in\Phi_1,\quad f\in Z. \tag{15}$$

Step 4. If $f\in Z$, then (6) yields $|f(x)|\leq M_1(x,f)$. A corresponding assertion holds for the vector-valued case. Now, (3) and (15) show that

$$\|f\|_{p,\varphi}\leq c\|\{F^{-1}\varphi_j Ff\}\|_{H_p(l_2)}\leq c'\|f\|_{H_p},\quad f\in Z. \tag{16}$$

Step 5. Let again $f \in Z$. (15) and the definition of $\|\cdot\|_{H_p(l_2)}$ yield

$$(17) \qquad \|f\|_{H_p} \leq c \|[\sum_{j=-\infty}^{\infty} (\sup\{|(F^{-1}\lambda(A_t.)\varphi_j Ff(x-y)| : d(y) \leq t\})^2]^{1/2}\|_{L_p}.$$

If $\lambda(A_t z)\varphi_j(z) \neq 0$ then $d(z) \geq c' 2^j$ and $d(A_t z) \leq c''$, where c' and c'' are positive numbers, independent of j. Hence, $t = d(A_t z)/d(z) \leq c 2^{-j}$, where c is independent of j. It follows that

$$\sup\{|(F^{-1}\lambda(A_t .)\varphi_j Ff)(x-y)| : d(y) \leq t\}$$
$$= \sup\{|\int_{R_n} (F^{-1}\lambda)(u)(F^{-1}\varphi_j Ff)(x-y-A_t u)du| : d(y) \leq t \leq c2^{-j}\}$$
$$\leq c' \sup\{|(F^{-1}\varphi_j Ff)(x-y-A_t u)|(1+|u|^a)^{-1} : u \in R_n,\ d(y) \leq t \leq c2^{-j}\},$$

where a is an arbitrary positive number (which will be chosen later on). Using $|A_2^j(y-A_t u)| \leq |A_{2^j} y| + |A_{2^j t} u| \leq c(1+|u|)$, if $d(y) \leq t \leq c2^{-j}$, then it follows that

$$(18) \qquad \sup\{|(F^{-1}\lambda(A_t)\varphi_j Ff)(x-y)| : d(y) \leq t\}$$
$$\leq c \sup|(F^{-1}\varphi_j Ff)(x-y-A_t u)|/(1+(A_{2^j}(y-A_t u)|^a) : u \in R_n,\ d(y) \leq t \leq c2^{-j}\}$$
$$\leq c \sup\{|(F^{-1}\varphi_j Ff)(x-z)|/(1+|A_{2^j} z|^a) : z \in R_n\} = c\varphi_j^* f(x).$$

Here $\varphi_j^* f(x)$ is essentially the tangential anisotropic maximal function from [10, p. 77]. (The use of rectangles in [10] instead of corridors is immaterial.) For the isotropic case, i. e. $a_1 = a_2 = \ldots = a_n \geq 1$, we refer also to [7, p. 125]. (17) and (18) yield

$$(19) \qquad \|f\|_{H_p} \leq c \|(\sum_{j=-\infty}^{\infty} |\varphi_j^* f(x)|^2)^{1/2}\|_{L_p}.$$

The maximal inequality in [10, p. 82], shows that the right-hand side of (19) can be estimated from above by $c' \|f\|_{p,\varphi}$. (Here the above number a must be chosen large enough!) Hence,

$$(20) \qquad \|f\|_{H_p} \leq c \|f\|_{p,\varphi}, \quad f \in Z, \quad \varphi \in \Phi_1,$$

where c is independent of f. (16) and (20) prove that the quasi-norms of H_p and F_p are equivalent to each other on Z. The proof is complete.

Remark 5. In the isotropic case, i. e. $a_1 = \ldots = a_n \geq 1$, the above proof can be shortened and simplified. For instance, $d(x)$ can be replaced by $|x|$. One obtains that the usual Hardy spaces H_p with $0 < p < \infty$ in the sense of C. Fefferman and E. M. Stein [3], coincide with the homogeneous isotropic spaces F_p. As mentioned in the introduction, this result is due to J. Peetre [7, 8]. But Peetre's proof is different. Beside multiplier theorems for the Hilbert space version of the usual Hardy spaces, he uses Chincin's inequality for the Rademacher functions.

3.2. Littlewood-Paley Theorem for BMO. If the numbers $a_1, a_2, \ldots, a_n$ have the above meaning, then we choose in this subsection $a_1 = a_2 = \ldots = a_n = 1$. This is the isotropic case (we remind of the fact that the spaces F_p, $\widetilde{F}_p$, and H_p depend on the choice of $a_1, \ldots, a_n$). In particular, $\widetilde{F}_\infty$ is an isotropic space.

Theorem 2. *It holds*

$$\text{BMO} = \tilde{F}_\infty. \tag{21}$$

Proof. Step 1. In Steps 2 and 3 of this proof we shall show that the dual space of F_1 is $\tilde{F}_\infty$,

$$(F_1)' = \tilde{F}_\infty. \tag{22}$$

Concerning the interpretation of dual spaces we remind of our convention, cf. the end of 2.5. Furthermore, we recall C. Fefferman's famous result

$$(H_1)' = \text{BMO}, \tag{23}$$

again with the above interpretation of dual spaces (isotropic case!), cf. [2, 3]. (21) follows from (22), (23), and (9).

Step 2. We prove (22). Let $f \in \tilde{F}_\infty$ be represented by (4), where we assume that $\|\{f_k\}\|_{L_\infty(l_2)} \leq 2\|f\|_{\tilde{F}_\infty}$. If $\varphi \in Z$ and if $\{\varphi_k\} \in \Phi_1$, then

$$|f(\varphi)| = \Big|\sum_{k=-\infty}^{\infty} f(F\varphi_k F^{-1}\varphi)\Big| = \Big|\sum_{k=-\infty}^{\infty} \sum_{l=k-1}^{k+1} (F^{-1}\varphi_l F f_l)(F\varphi_k F^{-1}\varphi)$$

$$= \Big|\sum_{|l-k|\leq 1} f_l(F\varphi_l\varphi_k F^{-1}\varphi)\Big|$$

$$\leq c\|\{f_l\}_{l=-\infty}^{\infty}\|_{L_\infty(l_2)}\|\{F\varphi_l\varphi_k F^{-1}\varphi\}_{|l-k|\leq 1}\|L_1(l_2) \leq c'\|f\|_{\tilde{F}_\infty}\|\varphi\|_{F_1}.$$

This proves $f \in (F_1)'$ and $\tilde{F}_\infty \subset (F_1)'$.

Step 3. Let $f \in (F_1)'$. If $\varphi \in F_1$, then, by definition, $\varphi \to \{F^{-1}\varphi_k F\varphi\}_{k=-\infty}^{\infty}$ is an one-to-one mapping from F_1 onto a subspace of $L_1(l_2)$. Hence, f can be interpreted as a functional on this subspace. By the Hahn-Banach theorem, f can be extended to a linear continuous functional on $L_1(l_2)$, where the norm of f is preserved. On the other hand, by standard arguments, one can show that $[L_1(l_2)]' = L_\infty(l_2)$, with the usual interpretation. Hence, we have

$$f(\varphi) = \int_{R_n} \sum_{k=-\infty}^{\infty} (F^{-1}\varphi_k F\varphi)(x) f_k(x)\,dx, \tag{24}$$

$$\|f\|_{(F_1)'} \sim \|\{f_k\}_{k=-\infty}^{\infty}\|_{L_\infty(l_2)}. \tag{25}$$

(24) yields

$$f(\varphi) = \sum_{k=-\infty}^{\infty} (F\varphi_k F^{-1} f_k)(\varphi). \tag{26}$$

(25) and (26) is a representation of type (4) (that F and F^{-1} changed their roles is immaterial). In particular $\|f\|_{\tilde{F}_\infty} \leq c\,\|f\|_{(F_1)'}$. Hence, $(F_1)' \subset \tilde{F}_\infty$. The proof is complete.

REFERENCES

1. A. P. Calderón, A. Torchinsky. Parabolic maximal functions associated with a distribution. I. II. *Advances in Math.*, **16**, 1975, 1—64.
2. C. Fefferman. Characterization of bounded mean oscillation. *Bull. Amer. Math., Soc.*, **77**, 1971, 587—588.
3. C. Fefferman, E. M. Stein. H^p spaces of several variables. *Acta Math.*, **129**, 1972, 137—193.
4. F. John, L. Nirenberg. On functions of bounded mean oscillation. *Comm. Pure Appl. Math.*, **14**, 1961, 415—426.
5. P. Kreé. Sur les multiplicateurs dans FL^p. *Ann. Inst. Fourier*, 16, 1966, 2, 31—89.
6. W. Littman, C. McCarthy, N. Riviere. L^p-multiplier theorems. *Studia Math.*, **30**, 1968, 193—217.
7. J. Peetre. On spaces of Triebel-Lizorkin type. *Ark. Math.*, **13**, 1975, 123—130.
8. J. Peetre. H_p-spaces. Lecture Notes Univ. Lund, Lund, 1974.
9. H. M. Reimann, T. Rychener. Funktionen beschränkter mittlerer Oszillation. Berlin, 1975.
10. H. Triebel. Fourier analysis and function spaces. Leipzig, 1977.

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