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ON CONVERGENT INTERPOLATORY PROCESSES*

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Summary. Necessary and sufficient conditions for convergence of interpolatory processes based on arbitrary nodes are investigated. Uniform and pointwise estimations are established. Particularly, a problem of G. Freud is solved.

1. Introduction and Preliminary Results. Considering any matrix $X=\{x_{k,n}=\cos\xi_{k,n}\}$ $(k=1,2,\dots,n;\ n=1,2,\dots)$ in $[-1,1]$ with

$$-1\leq x_{n,n}<x_{n-1,n}<\cdots<x_{2,n}<x_{1,n}\leq 1\ ;\ n=1,2,\dots. \tag{1.1}$$

We know that the Lagrange interpolatory procedure based on the nodes (1.1) cannot be uniformly convergent for all $f\in C$ (f is continuous on $[-1,1]$).

1.2. But if we won't restrict ourselves to the Lagrange interpolation we can state positive results. E. g., in 1943 P. Erdös [1] stated the following theorem.

Theorem A. *Let X be defined by (1.1). A necessary and sufficient condition that there should exist to every fixed $c>0$ a sequence of operators $\{L_n(f;c;x)=L_n(f;x)\}$ defined for $f\in C$ so that*

(i) $L_n(f;x)$ *is a polynomial of degree* $\leq n(1+c)$, $n=1,2,\dots,$

(ii) $L_n(f;x_{k,n})=f(x_{k,n})$; $k=1,2,\dots,n,$

(iii) $\lim\limits_{n\to\infty}\|L_n(f;x)-f(x)\|=0$ *for every* $f\in C$, *is that*

$$(\mathrm{E})\left\{\begin{array}{l}\text{if } n(\beta_n-\alpha_n)\to\infty,\ 0\leq\alpha_n<\beta_n\leq\pi \text{ then}\\ \overline{\lim\limits_{n\to\infty}}\ N_n(\alpha_n,\beta_n)/n(\beta_n-\alpha_n)\leq\pi^{-1} \text{ and } \underline{\lim\limits_{n\to\infty}}\ (\xi_{i+1,n}-\xi_{i,n})n>0;\ i=1,2,\dots,n-1.\end{array}\right.$$

Here $N_n(x;\alpha,\beta)=N_n(\alpha,\beta)$ stands for the number of the $\xi_{k,n}$ in $[\alpha,\beta]$ $\|g\|=\max\{|g(x)|:-1\leq x\leq 1\}$.

1.2. We wish to remark that $L_n(f;x)$ can be linear operators, too Namely we prove

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Theorem 1.1. *Considering an arbitrarily fixed X, the necessary and sufficient condition that there should exist to every fixed $c>0$ a sequence of linear operators $L_n(f; x)$ on C having* (i), (ii) *and* (iii) *is that* (E) *should be satisfied.*

1.3. In his paper [2] G. Freud investigated the rate of the convergence.

1.4. It is an interesting problem to state positive theorems for *arbitrary* systems of nodes which was investigated by J. Szabados [3].

1.5. Our aim is now to decide a *linear* operator-sequence where, as we shall see, the acting constant, in certain sense, will be the best possible. Further we deal with pointwise estimations, too.

2. New results. 2.1. Uniform approximation for arbitrary X. Using the above notations we can prove

Theorem 2.1. *Let us consider an arbitrary $X\subset[-1,1]$. Then for every fixed $c>0$ and $\varepsilon>0$ there exists a sequence of linear operators $L_n(f, c, \varepsilon; x)$ $=L_n(f; x)$ defined on C so that*

(a) *$L_n(f; x)$ is a polynomial of degree*

$$N\leqq\pi(1+c)/d_n;\ n=1,2,\dots,$$

(b) $L_n(f;\ x_{k,n})=f(x_{k,n});\ k=1,2,\dots,n,$

(c) $\|L_n(f;\ x)-f(x)\|=0(1)E_{[N(1-\varepsilon)]}(f);\ n=1,2,\dots;\ f\in C,$

where $d_n=d_n(X)=\min\{(\xi_{k+1,n}-\xi_{k,n}):\ 1\leqq k\leqq n-1\}$.
Generally the constant π cannot be changed for $\pi-\varrho$ $(\varrho>0)$.

2.2. Pointwise approximation for arbitrary X.

Theorem 2.2. *Let the matrix of nodes $X\subset[-1,1]$ be arbitrary, $r\geqq0$ fixed integer. Then for every fixed $c>0$ there exists a sequence of linear operators $K_n(f, c;\ x)=K_n(f;\ x)$ defined for every $f^{(r)}\in C$ for which* (a), (b) *and*

(d) $$|K_n(f;\ x)-f(x)|=0(1)(N^{-1}\sqrt{1-x^2}+N^{-2})^r\omega(f^{(r)};\ N^{-1}\sqrt{1-x^2}+N^{-2}),$$

$n\geqq n_0$ are valid.

2.3. A problem of G. Freud. Now we solve a generalized form of a question raised by G. Freud [4, Problem 10].

Theorem 2.3. *Let $X\subset[-1,1]$ and $r\geqq0$ be fixed. Then the necessary and sufficient condition that for every $c>0$ there should exist a sequence $\{A_n\}$ of linear polynomial operators defined for $f^{(r)}\in C$ such that*

$$\begin{cases}\deg A_n(f;\ x)\leqq n(1+c),\\ A_n(f;\ x_{k,n})=f(x_{k,n});\ k=1,2,\dots,\ n;\ n\geqq n_0,\\ |A_n(f;\ x)-f(x)|=0(1)(n^{-1}\sqrt{1-x^2}+n^{-2})^r\omega(f^{(r)};\ n^{-1}\sqrt{1-x^2}\\ \qquad\qquad +n^{-2});\ n\geqq n_0,\end{cases}$$

is that (E) should be satisfied.

2.4. Estimation of Hopenhaus — Telyakovski-type can be investigated, too.

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