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SOME REMARKS ON APPROXIMABLE ELEMENTS IN COUNTABLY MODULARED SPACES

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Summary. Let Ξ be an abstract set, and let $\widetilde{X}$ be a σ-algebra of subsets of the set Ξ. Moreover, let m be a nonnegative measure on $\widetilde{X}$. We take a family of modulars depending on a parameter $\bar{\varrho}=\varrho(\xi, x)$. Now, by means of this family, one may define the modulars: $\varrho_0(x)$, $\varrho_s(x)$ and $\varrho(x)$. If $\widetilde{\varrho}$ is any of the above modulars, then $X_{\widetilde{\varrho}}=\{x\in X: \widetilde{\varrho}(\lambda x)\to 0$ as $\lambda\to 0\}$ are modular spaces. In $X_{\widetilde{\varrho}}$ we may define an F-norm $\|\cdot\|_{\widetilde{\varrho}}$. Elements $x\in X$ belonging to $X_{\widetilde{\varrho}}$ will be called approximable by $\widetilde{\varrho}$.

In case when we take for X the space of all infinitely differentiable real-valued functions x in $]-\infty, \infty]$, we define the family of modulars $\bar{\varrho}_i(\xi, x)=\int_{-\infty}^{\infty}\varphi(\xi, |x^{(i-1)}(t)|)dt$. Next we define various modulars: $\varrho_{0i}(x)=\operatorname{supess}\bar{\varrho}_i(\xi, x)$, $\varrho_{si}(x)=\int_{\Xi}\bar{\varrho}_i(\xi, x)\,dm$ and $\varrho_i(x)=\int_{\Xi}p(\xi)\cdot(1+\bar{\varrho}_i(\xi, x))\cdot(\bar{\varrho}_i(\xi, x))^{-1}dm$, where $p(\xi)$ is measurable, $0<p(\xi)<\infty$, $\int_{\Xi}p(\xi)dm=1$.

We look for conditions that $\bar{\varrho}$ must satisfy in order that X_ϱ is contained in X_{ϱ_0}, and investigate connections between convergences in the sense of $\|\cdot\|_{\widetilde{\varrho}}$ and $\widetilde{\varrho}$.

In the paper [6] there was considered a family of modulars depending on a parameter.

Let X be a real or complex linear space and let Ξ be an abstract set, and let $\widetilde{X}$ be a σ-algebra of subsets of the set Ξ. Moreover, let m be a nonnegative measure on $\widetilde{X}$. We take $\bar{\varrho}:\Xi\times X\to]-\infty,\infty]$ satisfying the following conditions:

(1) $\bar{\varrho}$ is m-measurable in Ξ for every x,

(2) $\bar{\varrho}(\xi, 0)=0$ for m-almost every ξ,

(3) $\bar{\varrho}(\xi, x)=\bar{\varrho}(\xi, -x)$ for m-almost every ξ,

(4) $\bar{\varrho}(\xi, \alpha x+\beta y)\leq\bar{\varrho}(\xi, x)+\bar{\varrho}(\xi, y)$ for $\alpha, \beta\geq 0, \alpha+\beta=1$, $x, y\in X$ and for m-almost every ξ.

The map satisfying the foregoing conditions is called a family of pseudomodulars depending on a parameter. If $\bar{\varrho}$ satisfies the condition

(2') $\bar{\varrho}(\xi, x)=0$ in a set of ξ of positive measure implies $x=0$,

instead of the condition (2), then $\bar{\varrho}$ is called a family of modulars depending on a parameter, (see [4, 6]).

Let $\bar{\varrho}$ be a given family of modulars in X. By means of this family, one may define modulars

$$\varrho_0(x) = \operatorname{supess}_\xi \bar{\varrho}(\xi, x),$$

$$\varrho_s(x) = \int_\Xi \bar{\varrho}(\xi, x)dm$$

$$\varrho(x) = \int_\Xi p(\xi)\,(1+\bar{\varrho}(\xi, x))^{-1}\bar{\varrho}(\xi, x)\,dm,$$

where $p(\xi)$ is measurable, $0 < p(\xi) < \infty$, $\int_\Xi p(\xi)\,dm = 1$. In case when $\bar{\varrho}$ is a pseudomodular in X for every $\xi \in \Xi$, one may define also a modular $\varrho_u(x) = \sup_\xi \bar{\varrho}(\xi, x)$. If $\tilde{\varrho}$ is any of the modulars ϱ_0, ϱ_s, ϱ and ϱ_u, then $X_{\tilde{\varrho}} = \{x \in X : \tilde{\varrho}(\lambda x) \to 0$ as $\lambda \to 0\}$ is the respective modular space X_{ϱ_0}, X_{ϱ_s}, X_ϱ and X_{ϱ_u}.

Elements $x \in X$ belonging to $X_{\tilde{\varrho}}$ will be called approximable by $\bar{\varrho}$. In particular, elements $x \in X$ belonging to X_{ϱ_0} will be called uniformly approximable by $\bar{\varrho}$ (see also [3]). It is obvious that every uniformly approximable element belongs to the space of approximable elements X_ϱ, but the converse does not hold in general. Moreover, convergence in the space X_{ϱ_0} of uniformly approximable elements implies convergence in the space X_ϱ of approximable elements. The respective modular spaces $X_{\tilde{\varrho}}$ satisfy the following relations:

$$X_{\varrho_u} \subset X_{\varrho_0} \subset X_\varrho; \quad X_{\varrho_s} \subset X_\varrho; \quad \text{if } m(\Xi) < \infty, \text{ then } X_{\varrho_0} \subset X_{\varrho_s};$$

if Ξ consists of a countable number of pairwise disjoint atoms $A_1, A_2, \ldots$ and $\inf_k m(A_k) > 0$, then $X_{\varrho_s} \subset X_{\varrho_u}$; an element $x \in X$ belongs to X_ϱ, if and only if $\bar{\varrho}(\xi, \lambda x) \to 0$ as $\lambda \to 0$ m-almost every in Ξ.

Some generalizations of these problems we obtain, if we take a real linear space as X and if a family of measures $\mathfrak{M} = \{m_\eta\}$, $\eta \in H$, be given on a σ-algebra $\tilde{X}$ of subsets of an abstract set Ξ. We consider the family of modulars depending on a parameter $\bar{\varrho} = \bar{\varrho}(\xi, x)$ defined as above. By means of the family $\bar{\varrho}$ we define new pseudomodulars and modulars on X, and next we investigate connections between the respective modular spaces. In [7] two special cases were considered.

In the first case let us suppose that there is given a σ-algebra Y of subsets of the set H, and a measure n in Y. An extended real-valued function f in Ξ is said to be equal to 0 η-almost everywhere, if there exists a set $H_1 \subset H$ satisfying the condition $n(H \setminus H_1) = 0$ such that for every $\eta \in H_1$ there is a set $\Xi_1 \subset \Xi$ for which $m_\eta(\Xi \setminus \Xi_1) = 0$ and $\bar{\varrho}(\xi, x) = 0$ for all $\xi \in \Xi_1$. The following extended real-valued functionals are pseudomodulars in X:

$$\varrho_{00}(x) = \operatorname*{supess}_\eta \operatorname*{supess}_{\xi}{}_\eta\, \bar{\varrho}(\xi, x), \qquad \varrho_{ss}(x) = \int_H \int_\Xi \bar{\varrho}(\xi, x)dm_\eta dn,$$

$$\varrho_{0\sigma(\mathfrak{M})}(x) = \operatorname*{supess}_\eta \int_\Xi \bar{\varrho}(\xi, x)dm_\eta, \qquad \varrho^\varphi_{ss}(x) = \int_H \varphi\Big(\int_\Xi \bar{\varrho}(\xi, x)dm_\eta\Big)dn,$$

where φ is a φ-function in the sense of [5].

In the second case, if H is an abstract set, we say that an extended real-valued function f defined in Ξ is equal to 0 almost everywhere, if for each $\eta \in H$ there holds $m_\eta(\{\xi : f(\xi) \neq 0, \xi \in \Xi\}) = 0$.

The following extended-real valued functions are pseudomodulars in X:

$$\varrho_{u0}(x) = \sup_\eta \operatorname{supess}_\eta \bar{\varrho}(\xi, x), \quad \varrho_{u\sigma(\mathfrak{M})}(x) = \sup_\eta \int_\Xi \bar{\varrho}(\xi, x)\, dm_\eta.$$

Moreover, if $\eta_0 \notin H$, $H_0 = H \cup \{\eta_0\}$ is topological space and we denote

$$\overline{\lim_{\eta \to \eta_0}}\ g(\eta) = \inf_U \sup_{\eta \in U \setminus \{\eta_0\}} g(\eta),$$

where U runs over all neighbourhoods of η_0 in H_0, then

$$\varrho_{u\omega(\mathfrak{M})}(x) = \overline{\lim_{\eta \to \eta_0}} \int_\Xi \bar{\varrho}(\xi, x) dm_\eta$$

is also a pseudomodular on X.

In papers of J. Musielak and myself [6—11], the following problems were considered for a number of important special cases of $\bar{\varrho}$:

1) convergence in $X_{\tilde{\varrho}}$ in the sense of the F-norm $\|\cdot\|_{\tilde{\varrho}}$, that is $\|x_n - x\|_{\tilde{\varrho}} \to 0$ as $n \to \infty$, where $\|x\|_{\tilde{\varrho}} = \inf\{u > 0 : \tilde{\varrho}(x/u) \leq u\}$;

2) convergence in $X_{\tilde{\varrho}}$ in the sense of the modular $\tilde{\varrho}$, that is $\tilde{\varrho}(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for a certain $\lambda > 0$;

3) completeness of $X_{\tilde{\varrho}}$;

4) connections between the Orlicz classes $L^\psi(E, \tilde{E}, m)$ and the respective modular classes $X_{\tilde{\varrho}(\xi,\cdot)} = L^{\varphi(\xi,\cdot)}(E, \tilde{E}, m)$ for some φ-functions φ and ψ;

5) characterization of uniformly approximable elements among all approximable elements;

6) under which conditions on $\bar{\varrho}$ every approximable element is uniformly approximable.

First we consider the case when we take as X the space of all infinitely differentiable real-valued functions $x(t)$ in $]-\infty, \infty[$. Let φ be a convex φ-function in the sense [5]. We define a family of modulars by the formula $\bar{\varrho}_i(x) = \int_{-\infty}^{\infty} \varphi(|x^{(i-1)}(t)|)\, dt$.

The following may be proved.

Let $x_n \in X_{\varrho_0}$. Then $x_n \xrightarrow{\varrho} 0$ implies $x_n \xrightarrow{\varrho_0} 0$, and $\|x_n\|_\varrho \to 0$ implies $\|x_n\|_{\varrho_0} \to 0$.

Now we take a measure space (Ω, Σ, μ) with a finite measure μ. We define X as the space of all real-valued Σ-measurable functions over Ω, with equality μ-almost everywhere. Let (φ_i) be sequence of φ-functions, equicontinuous at zero. We take $\bar{\varrho}_i(x) = \int_\Omega \varphi_i(|x|)\, d\mu$. If we suppose that the following condition holds:

($\mathfrak{S}$) there exist positive constants $k, c, u_0 > 0$ and an index i_0 such that $\varphi_i(cu) \leq k\varphi_{i_0}(u)$ for all $u \geq u_0$ and all $i \geq i_0$, then we have the following

Theorem:

If ($\mathfrak{S}$) *holds and* $x_n \in x_\varrho$, *then* $x_n \in X_{\varrho_0}$, $x_n \xrightarrow{\varrho}$ *implies* $x_n \xrightarrow{\varrho_0} 0$, $\| x_n \|_\varrho \to 0$, *implies* $\| x_n \|_{\varrho_0} \to 0$, (compare [9]).

Under some additional assumption given in [9], condition ($\mathfrak{S}$) is also necessary. The results obtained in [9] are generalized in [10] to the case of modulars depending on a parameter, and in [11] for a special case of family of modulars $\bar{\varrho}$.

Now we define a map $\varrho_1 : \Xi x X \to [0, \infty]$ as follows:

$$\varrho_1(\xi, x) = \int_{-\infty}^{\infty} \varphi(\xi, |x(t)|)\, dt$$

and we put $\varrho_i(\xi, x) = \varrho_1(\xi, x^{(i-1)})$ for $i = 2, 3, \ldots$, where $x^{(i)}$ is the derivative of x of order i in the usual sense. Now, we may define various modulars in X:

$$\bar{\varrho}_{0i}(x) = \operatorname*{supess}_{\xi} \varrho_i(\xi, x),$$

$$\bar{\varrho}_{si}(x) = \int_{\Xi} \varrho_i(\xi, x)\, dm,$$

$$\bar{\varrho}_i(x) = \int_{\Xi} p(\xi)(1 + \varrho_i(\xi, x))^{-1} \varrho_i(\xi, x)\, dm,$$

where $p(\xi)$ is measurable, $0 < p(\xi) < \infty$, $\int_{\Xi} p(\xi)\, dm = 1$.

Moreover, let $\{m_\eta\}$, $\eta \in H$, be the family of nonnegative measures on $\widetilde{X}$, where H is an abstract set, then we write

$$\bar{\varrho}_{\sigma i}(x) = \sup_{\eta} \int_{\Xi} \varrho_i(\xi, x)\, dm_\eta .$$

If $\widetilde{\varrho_i}$ denotes one of these modulars $\bar{\varrho}_{0i}$, $\bar{\varrho}_{si}$ and $\bar{\varrho}_{\sigma i}$, then we write

$$\widetilde{\varrho}(x) = \sum_{i=1}^{\infty} 2^{-i} \widetilde{\varrho}_i(x)(1 + \widetilde{\varrho}_i(x))^{-1}, \quad \widetilde{\varrho}_0(x) = \sup_i \widetilde{\varrho}_i(x).$$

The respective modular spaces are denoted by $X_{\widetilde{\varrho}}$ and $X_{\widetilde{\varrho}_0}$, and as before are called the space of approximable elements and uniformly approximable elements, respectively. We may prove the following theorems about the space of approximable elements by the $\widetilde{\varrho}_i$.

Theorems: *If* $x_n \in X_{\widetilde{\varrho}_0}$, *then* $x_n \xrightarrow{\widetilde{\varrho}} 0$ *implies* $x_n \xrightarrow{\widetilde{\varrho}_0} 0$ *as* $n \to \infty$.

We give the proof of then theorems only in the case when $\widetilde{\varrho}$ and $\widetilde{\varrho}_0$ are defined by means of $\bar{\varrho}_{\sigma i}$ (compare [11]).

Proof. By the assumption $x_n \xrightarrow{\widetilde{\varrho}} 0$, there holds

$$\widetilde{\varrho}(\lambda_0 x_n) = \sum_{i=1}^{\infty} 2^{-i} \bar{\varrho}_{\sigma t}(\lambda_0 x_n)(1 + \bar{\varrho}_{\sigma i}(\lambda_0 x_n))^{-1} \to 0 \quad \text{as } n \to \infty$$

for a certain $\lambda_0 > 0$. Hence $\bar{\varrho}_{\sigma i}(\lambda_0 x_n) = \sup_\eta \int_{\Xi} \varrho_i(\xi, \lambda_0 x_n)\, dm_\eta \to 0$ as $n \to \infty$ for a certain $\lambda_0 > 0$, and in consequence, $\sup_\eta \int_{\Xi} \varrho_i(\xi, \lambda_0 x_n)\, dm_\eta < \infty$ for $n > N_i$, N_i independent of η. Thus, there exists a set $\Xi_1 \subset \Xi$ of measure $m_\eta(\Xi_1) = 0$, Ξ_1 independent of η, such that $\varrho_i(\xi, \lambda_0 x_n) < \infty$ for $n > N_i$, and $\xi \in \Xi \setminus \Xi_1$.

By the condition $x_n \in X_{\tilde{\varrho}_0}$, which means that $\tilde{\varrho}_0(\lambda x_n) = \sup_i \bar{\varrho}_{\sigma i}(\lambda x_n) \to 0$ as $\lambda \to 0$, we have

$$\bar{\varrho}_{\sigma i}(\lambda x_n) = \sup_\eta \int_\Xi \varrho_i(\xi, \lambda x_n) dm_\eta \to 0, \text{ and } \int_\Xi \varrho_i(\xi, \lambda x_n) dm_\eta \to 0$$

as $\lambda \to 0$ uniformly in i and for each n separately. Hence there are $\lambda_i' > 0$ independent of η such that $\int_\Xi \varrho_1(\xi, \lambda_i' x_n) dm_\eta < \infty$ for $n < N_i$. Now, if $\lambda_i = \min(\lambda_0, \lambda_i')$ we have $\int_\Xi \varrho_i(\xi, \lambda_i x_n) dm_\eta < \infty$ for $n < N_i$, and we conclude that there exists a set Ξ_2 dependent of η such that $m_\eta(\Xi_2) = 0$ and $\varrho_i(\xi, \lambda_i x_n) < \infty$ for $n < N_i$. Finally, we have $\varrho_i(\xi, \lambda_i x_n) < \infty$ for all n, i and for $\xi \in \Xi \setminus \Xi_0$, where $\Xi_0 = \Xi_1 \cup \Xi_2$ depends on η.

In [11] there is a lemma: If $x \in X$ and there exists a sequence of non-negative numbers (λ_i) such that $\varrho_i(\xi, \lambda_i x) < \infty$ for almost every $\xi \in \Xi$, then $\varrho_1(\xi, \lambda_1 x) \geqq \varrho_2(\xi, \lambda_1 x) \geqq \varrho_3(\xi, \lambda_1 x) \geqq \cdots$ almost everywhere in Ξ (compare [2]). By this lemma for m_η with fixed η, we have $\varrho_1(\xi, \lambda_1 x_n) \geqq \varrho_2(\xi, \lambda_1 x_n) \geqq \cdots$ and next

$$\int_\Xi \varrho_1(\xi, \lambda_1 x_n) dm_\eta \geqq \int_\Xi \varrho_2(\xi, \lambda_1 x_n) dm_\eta \geqq \cdots$$

Consequently, $\bar{\varrho}_{\sigma_1}(\lambda_1 x_n) \geqq \bar{\varrho}_{\sigma_2}(\lambda_1 x_n) \geqq \cdots$ But $\tilde{\varrho}_0(\lambda_1 x_n) = \sup_i \bar{\varrho}_{\sigma_i}(\lambda_1 x_n) \leqq \bar{\varrho}_{\sigma_1}(\lambda_1 x_n)$ $\to 0$ as $n \to \infty$, because $x_n \xrightarrow{\tilde{\varrho}} 0$ and this means that $x_n \xrightarrow{\tilde{\varrho}_0} 0$.

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