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APPROACH TO RATIONAL APPROXIMATION BY MEANS OF POLYNOMIAL APPROXIMATION

T. Winiarski

1. Introduction. The first problem which arises directly from the study of the rational approximation is the problem of choosing a suitable form of rational approximants. For example, using rational functions $r_{n\nu}(z)=(a_0z^n+\ldots+a_n)/(b_0z^\nu+\ldots+b_\nu)$; $n=1, 2, \ldots$, where ν is a fixed nonnegative integer, we can obtain (see e. g. [2], [9]) a characterization of meromorphic functions having at most ν poles by means of the rational approximation on compact sets $K\subset\mathbb{C}$.

Condition guaranteeing the single-valuedness of analytic functions have been given by A. A. Gončar in [1]. He used rational functions of the form $r_n(z)=(a_0z^n+\ldots+a_n)/(b_0z^n+\ldots+b_n)$; $n=1, 2, \ldots$. The functions $r_{n(s)}(z)=(a_0z^n+\ldots+a_n)(z-z_1)^{-k_1}\ldots(z-z_s)^{-k_s}$; $n=1, 2, \ldots$, where $k_1+\ldots+k_s\leqq n$ and s is a fixed nonnegative integer, can be used to the study of analytic functions having at most s singularities $a_1,\ldots,a_s$ with finite growth orders [4].

The natural problem is to give a characterization of orders of f at singularities $a_1,\ldots,a_s$ by means of the rational approximation. In order to do it we shall consider rational functions of the form

$$r_k(z)=p_{0k_0}(z)+p_{1k_1}(1/(z-a_1))+\ldots+p_{sk_s}(1/(z-a_s)),$$

where $k=(k_0,\ldots,k_s)\in N_0^{s+1}$ and p_{jk_j} are polynomials of degree $\leqq k_j$.

However, the principal subject of this paper is to give a method of conversion of the rational approximation (in $\mathbb{C}$) into the polynomial approximation of functions of several complex variables.

2. $\mathcal{A}$-Admissible Sets. Let $\mathcal{A}_j$; $j=0, 1, \ldots, s$ be the set of all analytic functions in $\mathbb{C}^{s+1}$ and satisfying conditions $f(w_0,\ldots,w_s)=h(w_j)$; $j=0,\ldots,s$ and $h(0)=0$; $j=1,\ldots s$. The spaces $\mathcal{A}_0,\ldots,\mathcal{A}_s$ are linear subspaces of $\mathcal{O}(\mathbb{C}^{s+1})$ such that $\mathcal{A}_i\cap\mathcal{A}_j=\{0\}$, $i\neq j$. We shall consider the direct sum $\mathcal{A}=\mathcal{A}_0\oplus\ldots\oplus\mathcal{A}_s$.

Let K be a compact set in $\mathbb{C}^{s+1}$ satisfying the following condition

(1) $$f\in\mathcal{A} \text{ and } f|_K=0 \text{ implies } f=0.$$

Writing $f\in\mathcal{A}$ in the form $f=f_0+\ldots+f_s$, $f_j\in\mathcal{A}_j$, we shall consider two norms in $\mathcal{A}$:

(2) $$\|f\|_K=\sup\{|f(w)|: w\in K\},$$

and

(3) $$|||f|||_K=\|f_0\|_K+\ldots+\|f_s\|_K.$$

As usually $\widehat{K}$ will denote the polynomial convex hull of K

$$\widehat{K}=\{w\in \mathbb{C}^{s+1}: |p(w)|\leq\|p\|_K \text{ for every polynomial } p\}.$$

If $K=K_0\times\ldots\times K_s$ has property (1), then the following theorem holds

Theorem 1. a) If $0\in \operatorname{int}\widehat{K}_j$; $j=1,\ldots s$, *then norms* (2), (3) *are equivalent and*

(4) $$\|f\|_K\leq|||f|||_K\leq(4s+1)\|f\|_K.$$

b) *If there exists* $j\in\{1,\ldots s\}$ *such that* $0\notin\widehat{K}_j$ *then the norms* (2), (3) *are not equivalent.*

Proof. Fix $\mathring{w}_{<j>}=(\mathring{w}_0,\ldots,\mathring{w}_{j-1},\mathring{w}_{j+1},\ldots,\mathring{w}_s)K_0\times\ldots\times K_{j-1}\times K_{j+1}\times\ldots\times K_s$ and define $\psi(\cdot,\mathring{w}_{<j>}):\mathbb{C}\ni\lambda\longrightarrow(\mathring{w}_0,\ldots,\mathring{w}_{j-1},\lambda,\mathring{w}_{j+1},\ldots,\mathring{w}_s)\in\mathbb{C}^{s+1}$ and $\widetilde{K}_j=\psi(K_j,\mathring{w}_{<j>})$. Let $A=A(\mathring{w}_{<j>})=f_0(\mathring{w}_0)+\ldots+f_{j-1}(\mathring{w}_{j-1})+f_{j+1}(\mathring{w}_{j+1})+\ldots+f_s(\mathring{w}_s)$. Then

$$\|f\|_{\widetilde{K}_j}=\sup\{|f_j(\lambda)+A|:\lambda\in K_j\}=\|f_j+A\|_{K_j}=\|f_j+A\|_{\widehat{K}_j}\leq\|f\|_K.$$

Hence

(5) $$\|f_j\|_K\leq\|f\|_K+|A|.$$

It follows from the definition of the space $\mathcal{A}_j$ that $f_j(\lambda)=\lambda g_j(\lambda)$, where g_j is an entire function in $\mathbb{C}$. Let $r>0$ be such that $S(r)=\{\lambda\in\mathbb{C}:|\lambda|=r\}\subset\widehat{K}_j$. Then $\|f\|_K\geq\|f_j+A\|_{\widetilde{K}_j}\geq\|f_j+A\|_{S(r)}=r\,\|g_j(\lambda)+A/\lambda\|_{S(r)}$.

Hence

(6) $$\|g_j(\lambda)+A/\lambda\|_{S(r)}\leq r^{-1}\|f\|_K.$$

From (6) and by the residuum theorem

$$|A|=\Big|\frac{1}{2\pi i}\int_{S(r)}(g_j(\lambda)+A/\lambda)\,d\lambda\Big|\leq\|f\|_K.$$

Therefore

(7) $$\|f_j\|_K=\|f_j\|_{K_j}\leq 2\,\|f\|_K;\ j=1,\ldots,s.$$

Since $f_0=f-(f_1+\ldots+f_s)$, so by the triangle inequality

(8) $$\|f_0\|_{K_0}\leq\|f\|_K+\|f_1\|_K+\ldots+\|f_s\|_K\leq(2s+1)\|f\|_K.$$

Therefore, the second inequality of (4) is true.

The first inequality of (4) is a trivial consequence of the triangle inequality.

If $0\notin\widehat{K}_j$ then by Runge's Theorem there are sequences $\{f_{0\nu}\}$, $\{f_{j\nu}\}$ such that

1) $f_{k\nu}\in\mathcal{A}_k$, $k=0, j$, $\nu=1, 2,\ldots,$

2) $\lim\limits_{\nu\to\infty}\|f_{0\nu}-1\|_{K_0}=0$, $\lim\limits_{\nu\to\infty}\|f_{j\nu}+1\|_{K_j}=0.$

If we take $F_\nu = f_{0\nu} + f_{j\nu}$, $\nu = 1, 2, \ldots$, then we obtain $\lim_{\nu\to\infty} \| F_\nu \|_K = 0$ and $\lim_{\to\infty} ||| F_\nu |||_K = 2$ which contradicts the equivalency of the norms (2), (3).

A class $\mathfrak{M}^*$ of compact sets $K = K_0 \times \ldots \times K_s$ will be called $\mathcal{A}$-admissible class if for every compact $K \in \mathfrak{M}^*$ norms (2), (3) are equivalent. Each set $K \in \mathfrak{M}^*$ will be called $\mathcal{A}$-admissible set.

2. Polynomial Approximation on $\mathcal{A}$-Admissible Sets. Let $\mathcal{O}_k$, $k = (k_0, \ldots, k_s)$ be the set of all polynomials of $s+1$ complex variables of degree $\leq k_j$ with respect to the j-th variable, respectively. Let K be a compact set in $\mathbb{C}^{s+1}$ and let $f: K \to \mathbb{C}$ be continuous in K.

Write $\mathcal{E}_k^*(f, K) = \inf\{ \| f - p \|_K : p \in \mathcal{O}_k \}$ and $E_k^*(f, K) = \inf\{ \| f - p \|_K : p \in \mathcal{O}_k \cap \mathcal{A} \}$.

Now, suppose that $f = f_0 + \ldots + f_s \in \mathcal{A}$ and let $K = K_0 \times \ldots \times K_s$ be an $\mathcal{A}$-admissible set in $\mathbb{C}^{s+1}$ and write

$$d_{jk_j}(f, K) = \inf\{ \| f - p \|_K : p \in \mathcal{O}_k \cap \mathcal{A}_j \} \quad \text{and} \quad M_k^*(f, K) = \inf\{ ||| f - p |||_K : p \in \mathcal{O}_k \cap \mathcal{A} \}.$$

Given a compact set $K \subset \mathbb{C}^p$ and a subset Ω of $\mathbb{C}^q$ we denote by $(\Phi_\lambda)_{\lambda \in \Omega}$ the family of functions in K such that the mapping $\Omega \times K \ni (\lambda, z) \to \Phi_\lambda(z) \in \mathbb{C}$ is continuous. We shall use the following

Lemma 1. *If $\{\overset{\nu}{\lambda}\}$, $\{\overset{\nu}{z}\}$ are two sequences such that*

1^0 $(\overset{\nu}{\lambda}, \overset{\nu}{z}) \in \Omega \times K$, $\nu = 0, 1, \ldots$; 2^0 $\lim_{\nu\to\infty} \overset{\nu}{\lambda} = \overset{\circ}{\lambda}$, $\lim_{\nu\to\infty} \overset{\nu}{z} = \overset{\circ}{z}$;

3^0 $\| \Phi_{\overset{\nu}{\lambda}} \|_K = | \Phi_{\overset{\nu}{\lambda}}(\overset{\nu}{z}) |$; $\nu = 1, 2, \ldots$, *then* $\| \Phi_{\overset{\circ}{\lambda}} \|_K = | \Phi_{\overset{\circ}{\lambda}}(\overset{\circ}{z}) |$.

We can now state the following theorem.

Theorem 2. *If $K = K_0 \times \ldots \times K_s$ is an $\mathcal{A}$-admissible set then*

$$(9) \quad (4s+1)^{-1} \sum_{j=0}^{s} d_{jk_j}(f, K) \leq \mathcal{E}_k^*(f, K) \leq E_k^*(f, K) \leq M_k^*(f, K) \leq \sum_{j=0}^{s} d_{jk_j}(f, K).$$

Proof. Fix polynomial $P_k(w_0, \ldots, w_s) = \sum_{\alpha \leq k} a_\alpha w^\alpha$ (where $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s$, $\alpha \leq k \Leftrightarrow \alpha_j \leq k_j$, $j = 0, \ldots, s$) and rewrite it in the form

$$P_k(w_0, \ldots, w_s) = Q_{k_0}(w_0; w_{<0>}) + Q_{k_1}(w_1; w_{<0,1>}) + \ldots +$$

$Q_{k_s}(w_s; w_{<0,\ldots,s>})$, where $Q_{k_j}(\cdot; w_{<0,\ldots,j>})$ are polynomials in the j-th, variable w_j of degree $\leq k_j$ with coefficients depending only on $w_{j+1}, \ldots, w_s$, $j = 0, \ldots, s$.

Let us write $\Phi_\lambda(w) = f_0(w_0) - Q_{k_0}(w_0; \lambda) + f_1(w_1) - Q_{k_1}(w_1; \lambda_{<1>}) + \ldots + f_s(w_s) - Q_{k_s}(w_s; \lambda_{<1,\ldots,s>})$ for $\lambda = (\lambda_1, \ldots, \lambda_s) \in K_1 \times \ldots \times K_s$, $w \in \mathbb{C}^{s+1}$. We see at once that

1) $\Phi_\lambda \in \mathcal{A}$ for $\lambda \in K_{<0>} = K_1 \times \ldots \times K_s$,

2) $\Phi_{w_{<0>}}(w) = f(w) - P_k(w)$,

3) $\| \Phi_\lambda \|_K \geq (4s+1)^{-1} ||| \Phi_\lambda |||_K \geq (4s+1)^{-1} M_k^*(f, K) = (4s+1)^{-1} \sum_{j=0}^{s} d_{jk_j}(f, K)$.

Let $\overset{\circ}{w}$ be such a point of K that $\| f - P_k \|_K = | f(\overset{\circ}{w}) - P_k(\overset{\circ}{w}) | = | \Phi_{\overset{\circ}{w}_{<0>}}(\overset{\circ}{w}) |$. We define by induction sequences $\{\overset{\nu}{\lambda}\}, \{\overset{\nu}{w}\}$. We put $\overset{1}{\lambda} = \overset{\circ}{w}_{<0>}$. Suppose we have defined $\overset{\nu-1}{\lambda}$. Then there exists $\overset{\nu-1}{w} \in K$ such that $\| \Phi_{\overset{\nu-1}{\lambda}} \|_K = | \Phi_{\overset{\nu-1}{\lambda}} (\overset{\nu-1}{w}) |$ and we define $\overset{\nu}{\lambda} = \overset{\nu-1}{w}_{<0>}$.

By choosing, if necessary, a convergent subsequence we can assume that $\lim_{\nu\to\infty} \overset{\nu}{\lambda} = \widetilde{\lambda} = \widetilde{w}_{<0>}$, $\lim_{\nu\to\infty} \overset{\nu}{w} = \widetilde{w}$. By virtue of Lemma 1 we have $\| \Phi_{\widetilde{w}_{<0>}} \|_K = | \Phi_{\widetilde{w}_{<0>}} (\widetilde{w}) |$. Hence, since λ and P_k are arbitrary (in 3), it follows that $(4s+1)^{-1} M_k^*(f, K) \leq \mathcal{E}_k^*(f, K)$. The rest inequalities (9) are easy to check.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be an entire function. For each $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ define $\mathfrak{M}_f(r) = \sup\{ |f(z_1, \ldots, z_n)| : |z_j| = r_j,\ j = 1, \ldots, n\}$.

Let $B(f)$ be the interior of the set of all $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ such that a constant r_a exists such that

$$\log^+ \mathfrak{M}_f(r) \leq r_1^{a_1} + \ldots + r_n^{a_n}$$

for all $r = (r_1, \ldots, r_n)$ with $|r| > r_a$, where $|r| = r_1 + \ldots + r_n$. If $B(f) \neq \emptyset$, then f is said to have finite order.

Let $S(f) = \partial B(f)$ be the boundary of $B(f)$. Then each $r \in S(f)$ is called a system of associated orders of f. We call

$$\varrho_j = \inf\{ a_j : (a_1, \ldots, a_n) \in B(f)\}$$

the j-th order of f.

Now, let $f = f_0 + \ldots + f_s$ be a function of the space $\mathcal{A}$. By Theorem 3.1.2 in [8] we see at once that ϱ_j is the j-th order of f if and only if ϱ_j is the order of f_j, $j = 1, \ldots, s$.

Assume that f_j is not polynomial, for $j = 0, \ldots, s$. Put $\Lambda_j = \{k : d_{jk_j}(f, K) = \max\{d_{k_0}, \ldots, d_{sk_s}\}\}$, and observe that $\pi_j(\Lambda_j) = \mathbb{N}$, where $\pi_j : \mathbb{R}^{s+1} \ni x = (x_0, \ldots, x_s) \to x_j \in \mathbb{R}$ is the j-th projection. Moreover, if $K = K_0 \times \ldots \times K_s$ is an $\mathcal{A}$-admissible set then $(4s+1)^{-1} d_{jk_j}(f, K) \leq \mathcal{E}_k^*(f, K) \leq (s+1) d_{jk_j}(f, K)$ for $k \in \Lambda_j$. Hence by standard reasoning we obtain

$$\limsup_{\substack{k\to\infty \\ k \in \Lambda_j}} (k_j \ln k_j / (-\ln \mathcal{E}_k^*(f, k))) = \limsup_{k_j \to\infty} (k_j \ln k_j / (-\ln d_{jk_j}(f, K))).$$

Then, see e. g. [6] Theorem 4.2, the following theorem holds.

Theorem 3. *If the transfinite diameter of K_0 is positive and $K = K_0 \times \ldots \times K_s \in \mathfrak{W}^*$ then a nonnegative number ϱ_j is the j-th order of f if and only if*

$$\varrho_j = \limsup \{ k_j \ln k_j / (-\ln \mathcal{E}_k^*(f, K)) : k \to \infty,\ k \in \Lambda_j,\ j = 0, \ldots, s\}.$$

On account of Theorem 2 we see at once that $\limsup_{k\to\infty} (k_j \ln k_j / (-\ln \mathcal{E}_k^*(f, K)) = \infty$. So, the assumption $k \in \Lambda_j$ cannot be omitted.

3. Rational Approximation on $\mathfrak{R}$-Admissible Sets. Given an arbitrary sequence of s complex numbers $a_1, \ldots, a_s$ consider the space $H = H(a_1 \ldots, a_s)$ of all analytic functions in $\mathbb{C} - \{a_1, \ldots, a_s\}$. The space H can be written as the direct sum $H = H_0 \oplus \ldots, \oplus H_s$, where H_0 is the space of all entire functions and H_j is the space of all functions f analytic in $\mathbb{C} - \{a_j\}$ such that $f(\infty) = 0$.

Let K be such a compact set that 1° $a_j \notin K$; $j=1,\ldots,s$, and 2° $f \in H$ and $f|_K=0 \Longrightarrow f=0$. Given a function $f=f_0+\ldots+f_s$ of the space H we shall consider two norms

$$\|f\|_K=\sup\{|f(z)|: z\in K\} \text{ and } |||f|||_K=\|f_0\|_K+\ldots+\|f_s\|_K.$$

For each integer $j \in \{0,\ldots,s\}$, let us denote by $\overline{H}_j$ the closure of H_j in $\mathfrak{C}(K)$ ($\mathfrak{C}(K)$ is the Banach space of all continuous functions on K). It follows from Runge's Theorem that $\overline{H}_i \cap \overline{H}_j=\{0\}$ $i\neq j$, if and only if the following condition is satisfied

Condition W. The complement $C-K$ of the set K consists at least of s different bounded components $D_1,\ldots,D_s$ such that $a_j \in D_j$, $j=1,\ldots,s$.

Suppose that Condition W is satisfied. Then we can define $\widetilde{H}=\overline{H}_0\oplus\ldots\oplus\overline{H}_s$. The space $\widetilde{H}$ is a normed subspace of $\mathfrak{C}(K)$ but in general it is not a Banach subspace of $\mathfrak{C}(K)$. Condition W is only a sufficient one (this follows from results of [3]).

A class $\mathfrak{M}=\mathfrak{M}(a_1,\ldots a_s)$ of compact sets in $\mathbb{C}$ will be called R-admissible class (rational admissible class) if for every compact $K\in\mathfrak{M}$ the space $\widetilde{H}$ is complete. Each set $K\in\mathfrak{M}$ will be called R-adimissible set. If e. g. D is a bounded domain in C the complement $C-\overline{D}$ of which consists of at least s bounded components $D_1,\ldots,D_s$ such that $a_j\in D_j$, then $K=\overline{D}\in\mathfrak{M}$.

Let $\varphi_j:\overline{C}\to\overline{C}$; $j=0,\ldots,s$ be defined as follows: $\varphi_0(z)=z$, $\varphi_j(z)=(z-a_j)^{-1}$, $j=1,\ldots,s$. Given any set $K\in\mathfrak{M}$ we put $K_j=\varphi_j(K)$, $j=0,\ldots,s$, and $\widetilde{K}=K_0\times\ldots\times K_s\subset\mathbb{C}^{s+1}$.

With the previous notations it is easy to see that for $\widetilde{K}$ inequalities (4) hold true. Let $F(w_0,\ldots,w_s)=F_0(w_0)+\ldots+F_s(w_s)$ be a function of the space $\widetilde{\mathcal{A}}(a_1,\ldots,a_s)$ and define a linear mapping

$$\Phi:\widetilde{\mathcal{A}}\longrightarrow\widetilde{H}$$

as follows

$$\Phi F(z)=F_0(z)+F_1(1/(z-a_1))+\ldots+F_s(1/(z-a_s)).$$

Since there exists $\mathring{z}\in K$ such that

$$\|\Phi F\|_K=|F_0(\mathring{z})+F_1(1/(\mathring{z}-a_1))+\ldots+F_s(1/(\mathring{z}-a_s))|$$

we have $\|\Phi F\|_K\leq\|F\|_{\widetilde{K}}$. Therefore, the mapping Φ is a linear continuous bijection of the Banach space $\widetilde{\mathcal{A}}$, $\widetilde{H}$. Thus, by a well-known Banach Theorem (see e. g. [10], Th. 2. 1, Chap. III) Φ and Φ^{-1} are continuous.

Finally we have the following

Theorem 4. *If $K\in\mathfrak{M}$ then there exists $\delta>0$ such that*

$$\delta\|F\|_{\widetilde{K}}\leq\|\Phi F|_K\leq\|F\|_{\widetilde{K}}. \tag{10}$$

We see at once that every rational function w with poles at $a_1,\ldots,a_s$ can be written in the form

$$w(z)=p_{0k_0}(z)+p_{1k_1}(1/(z-a_1))+\ldots+p_{sk_s}(1/(z-a_s)), \tag{11}$$

where p_{jk_j} being polynomials of degree k_j, $j=0,\ldots,s$ and $p_{jk_j}(0)=0$; $j=1,\ldots,s$.

Let $k=(k_0,\dots,k_s)$ be a sequence of $s+1$ non-negative integers. Define

$$\mathcal{E}_k(f, K)=\inf\{\|f-w\|_K\},$$

where the infimum is taken over all rational functions of the form (11).

On account of Theorem 3 and inequalities (10) we therefore obtain the following

Theorem 5. *If $K\in\mathfrak{M}$ and $f=f_0+\dots+f_s\in\widetilde{H}$ is a such function that f_j is not rational function for $j=0,\dots,s$ then there exist $\Lambda_j\in\mathbb{N}^{s+1}$; $j=0,\dots,s$, such that a non-negative integer ϱ_j is the order of f at aj if and only if* $\varrho_j=\limsup\limits_{\substack{k\to\infty\\ k\in\Lambda_j}} k_j\ln k_j/(-\ln\mathcal{E}_k(f, K))$.

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Krakow Poland *Received September 27, 1977*