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INTERPOLATION BY COMPLEX CUBIC SPLINES

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Summary. The aim of the paper is the application of spline theory to the approximation of a function analytic interior to a rectifiable Jordan curve and continuous in the corresponding closed region. This work is concerned with cubic splines in the complex variable z and it serves to estimate the convergence of the complex cubic spline and its derivatives for the situation in which the approximated function is of class C^a; $a=0, 1, 2,$ or 3 on the boundary.

The results obtained are a generalization from theorems of J. H. Ahlberg, E. N. Nilson and J. L. Walsh (1967).

Introduction. The purpose of this paper is to give theorems on interpolation to analytic functions in a Jordan region D and continuous on its boundary Γ by complex cubic splines.

Let Γ be a rectifiable Jordan curve and let $t_1, t_2, \ldots, t_N$ be points on Γ arranged in counterclockwise order, separating Γ into arcs Γ_j, $j=1,\ldots, N$ with Γ_j the arc from t_{j-1} to t_j; $t_0=t_N$. Denote this partition of Γ by Δ. The function $s_\Delta \in C^{m-1}(\Gamma\setminus\{t_0\})$ is called a *complex spline of degree m with respect to the partition* Δ, where $m\geqq 1$, if it is on each arc Γ_j a polynomial of degree at most m in the complex variable z.

The function S_Δ defined by the Cauchy integral

$$(1) \qquad S_\Delta(z)=(1/2\pi i)\int_\Gamma (t-z)^{-1}s_\Delta(t)dt, \quad z\in D$$

is said to be an *analytic spline of degree m with respect to the function* s_Δ. Because the function s_Δ satisfies Lipschitz's condition on Γ so we can define the function S_Δ on Γ as the limiting value (cf. [3], [5]) for approach from within

$$(2) \qquad S_\Delta(t)=(1/2)s_\Delta(t)+(1/2\pi i)\int_\Gamma (\tau-t)^{-1}s_\Delta(\tau)d\tau, \quad t\in\Gamma,$$

where the integral is interpreted as Cauchy principal values.

Analytic functions defined in this way were introduced by J. H. Ahlberg, E. N. Nilson and J. L. Walsh in [1].

The modulus of continuity of the function f defined in the set D_f is defined by the formula

$$(3) \qquad \omega(f, h)=\sup\{|f(t_2)-f(t_1)|: 0<|t_2-t_1|\leqq h; \quad t_1, t_2\in D_f\}.$$

A Jordan curve Γ is called a curve of class $S_{h,\lambda}$, if for every two points $t_1, t_2 \in \Gamma$, separating Γ into arcs Γ_1 and Γ_2, such that $|t_2-t_1| \leq h$ $\min(|\Gamma_1|, |\Gamma_2|) \leq \lambda |t_2-t_1|$, where $|\Gamma_j|$ is the length of Γ_j. Further we shall investigate complex cubic splines defined on curves of class $S_{h,\lambda}$ with $\lambda<2$, $3\|\Delta\| \leq h$, where $\|\Delta\| = \max\{|t_j-t_{j-1}| : j=1,\ldots,N\}$. Let g be a given function on Γ. For convenience we introduce the following notations: $g_j^{(r)} = g^{(r)}(t_j)$, $g_{j+}^{(r)} = g^{(r)}(t_{j+}) = \lim g^{(r)}(t)$, where $t \to t_j$, $t \in \Gamma_{j+1}$, $g_{j-}^{(r)} = g^{(r)}(t_{j-}) = \lim g^{(r)}(t)$, where $t \to t_j$, $t \in \Gamma_j$, $h_j = t_j - t_{j-1}$, $\lambda_j = h_{j+1}/(h_j+h_{j+1})$, $\mu_j = 1-\lambda_j$, $|||\Delta||| = \min\{|t_j-t_{j-1}| : j=1,\ldots,N\}$, $K_N = \|\Delta\|/|||\Delta|||$. For a given cubic spline s_Δ with respect to the partition Δ we set: $M_j = s''_\Delta(t_j)$, $m_j = s'_\Delta(t_j)$, $j=1, \ldots, N$, $m_{0+} = s'_\Delta(t_{0+})$, $m_{0-} = s'_\Delta(t_{0-})$. For $t \in \Gamma_j$ we can write the function s_Δ in one of two forms:

$$s_\Delta(t) = M_{j-1}(t_j-t)^3/6h_j + M_j(t-t_{j-1})^3/6h_j + (f_{j-1}/h_j - M_{j-1}h_j/6)(t_j-t) + (f_j/h_j - M_jh_j/6)(t-t_{j-1}) \tag{4}$$

or

$$s_\Delta(t) = h_j^{-2} m_{j-1}(t_j-t)^2(t-t_{j-1}) - h_j^{-2} m_j (t-t_{j-1})^2(t_j-t) + h_j^{-3} f_{j-1}(t_j-t)^2[2(t-t_{j-1})+h_j] + h_j^{-3} f_j (t-t_{j-1})^2[2(t_j-t)+h_j]. \tag{5}$$

Let s_f be a cubic spline with respect to the partition Δ satisfying the following conditions:

a) If $f \in C(\Gamma)$, then $s_f \in C^2(\Gamma)$ and $s_f(t_j) = f_j$, $j=1,\ldots,N$;

b) If $f \in C^1(\Gamma \setminus \{t_0\})$ and the functions f and f' have one-sided limits at the point t_0, then $s_f \in C^2(\Gamma \setminus \{t_0\})$, $s''_f \in C(\Gamma)$, $s_f(t_j) = f_j$, $j=1,\ldots,N-1$, $s_f(t_{0+}) = f_{0+}$, $s_f(t_{0-}) = f_{0-}$ and $m_{0-} - m_{0+} = f'_0 - f'_{0+}$.

Identical as in [1] we obtain from (4) the following system of equalities:

$$\begin{aligned} &\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = 6f[t_{j-1}, t_j t_{j+1}], \quad j=1,\ldots,N-1 \\ &\mu_N M_{N-1} + 2M_N + \lambda_N M_1 = 6(h_N+h_1)^{-1}\{(f'_0 - (f_{0-} - f_{N-1})/h_N) \\ &\qquad - (f'_{0+} - (f_1 - f_{0+})/h_1)\}, \end{aligned} \tag{6}$$

where $f[t_{j-1}, t_j, t_{j+1}]$ is the second divided difference (cf. [2, 7]) involving functional values f_{j-1}, f_j, f_{j+1} at the points t_{j-1}, t_j, t_{j+1}. The existence of the spline s_f on Γ assuming prescribed values f_j at the points t_j rests upon the possibility of solving the system (4) for the quantities M_j. Now the coefficient matrix A has dominant main diagonal provided that, for each $j (j=1,\ldots,N)$, $|h_j| + |h_{j+1}| < 2|h_j + h_{j+1}|$. Because the curve Γ is of class $S_{h,\lambda}$ with $3\|\Delta\| \leq h$, $\lambda < 2$ so the above condition is satisfied. Then the function s_f exists and is unique for arbitrary complex values $f_1, \ldots, f_N$.

Analogously as (5) we obtain the following system of equalities:

$$\begin{aligned} &\lambda_j m_{j-1} + 2m_j + \mu_j m_{j+1} = 3(\mu_j(f_{j+1}-f_j)/h_{j+1} + \lambda_j(f_j - f_{j-1})/h_j) \quad j=1,\ldots,N, \\ &\lambda_{N-1} m_{N-2} + 2m_{N-1} + \mu_{N-1} m_{0+} = 3(\mu_{N-1}(f_{0-} - f_{N-1})/h_N \\ &\qquad + \lambda_{N-1}(f_{N-1} - f_{N-2})/h_{N-1}) + \mu_{N-1}(f'_{0+} - f'_{0-}) \\ &\lambda_N m_{N-1} + 2m_{0+} + \mu_N m_1 = 3(\mu_N(f_1 - f_{0+})/h_1 + \lambda_N(f_{0-} - f_{N-1}/h_N) + \lambda_N(f'_{0+} - f'_{0-}) \\ &m_{0-} = m_{0+} + (f'_{0-} - f'_{0+}). \end{aligned} \tag{7}$$

Let Γ be a smooth Jordan curve (i. e., it has a continuously turning tangent), $\{\Delta_N\}_{N=1}^{\infty}$ a given sequence of partitions of Γ such that $\lim_{N\to\infty}\|\Delta_N\|=0$, f a given function defined on Γ, $\|f\|=\sup_{z\in\Gamma}|f(z)|$, $s_{f,N}$ a complex cubic spline with respect to the partition Δ_N of interpolation to f on Γ (i. e. $s_{f,N}(t_j)=f_j$, $j=1,\dots,N$) and $S_{f,N}$ an analytic spline with respect to the function $s_{f,N}$.

Under the above assumptions J. H. Ahlberg, E. N. Nilson and J. L. Walsh [1] proved the following theorems:

Theorem A. *If $f\in C^r(\Gamma)$, $r=0, 1, 2$ or 3 (for $r=0$ or 3 we assume that $K_N\leq K<\infty$), then $s_{f,N}^{(r)}(t)$ converges uniformly on Γ to $f^{(r)}(t)$. If $f^{(r)}$ satisfies a Hölder condition on Γ of order $\beta(0<\beta\leq 1)$, then*

$$\|s_{f,N}^{(i)}-f^{(i)}\|=O(\|\Delta_N\|^{r-i+\beta}),\quad 0\leq i\leq r. \tag{8}$$

Theorem B. *If the function f is analytic in D, $f^{(r)}$ $r=0, 1, 2$ continuous in $\bar{D}=D\cup\Gamma$ and satisfies a Hölder condition on Γ of order $\beta(0<\beta\leq 1)$, then for any β', $0<\beta'<\beta$*

$$\|S_{f,N}^{(i)}-f^{(i)}\|=o(\|\Delta_N\|^{r-i+\beta'}),\quad 0\leq i\leq r, \tag{9}$$

and for $i=1$, $r=2$ we have no restriction on the values K_N.

In this paper we shall give a simpler proof of some part of Theorem A and an estimate of the differences $f^{(i)}-s_{f,N}^{(i)}$ and $f^{(i)}-S_{f,N}^{(i)}$ depending on the modulus of continuity of the function $f^{(r)} i\leq r$ on Γ. Then we shall conclude that in Theorem B $\beta'=\beta$ for $i=1$ and 2.

2. Convergence on the Boundary. Under the above assumptions we shall prove the following

Theorem 1. *If $f\in C^p(\Gamma)$, $p=0, 1, 2$ or 3, then*

$$\|f^{(i)}-s_{f,N}^{(i)}\|\leq\begin{cases}A_{pi}(\lambda)\|\Delta_N\|^{p-i}\omega(f^{(p)},\|\Delta_N\|), & i\leq p<3,\ p\neq 0\\ A_{pi}(\lambda)K_N\|\Delta_N\|^{p-i}\omega(f^{(p)},\|\Delta_N\|), & i\leq p,\ p=0.3,\end{cases} \tag{10}$$

where A_{ik} is a constant depending only on λ.

Now Theorem A is a conclusion from Theorem 1.

Proof. 1) f' is continuous on Γ except at most finite number of points at which it has one-sided limits.

We can write the third equality of (7) as follows:

$$\begin{aligned}\lambda_N(m_{N-1}-f'_{N-1})+2(m_{0+}-f'_{0+})+\mu_N(m_1-f'_1)=\lambda_N(e_N-f'_{N-1})\\ +\mu_N(e_1-f'_1)+2[\mu_N(e_1-f'_{0+})+\lambda_N(e_N-f'_{0-})\\ +\lambda_N(f'_{0-}-f'_{0+}),\quad \text{where}\quad e_j=(f_j-f_{j-1})/h_j\end{aligned} \tag{11}$$

and if f'_j does not exist, then we set $f'(t_{j+})$. For remaining m_j we obtain analogous equalities. Further

$$f'(t)-e_j=h_j^{-1}\int_{t_{j-1}}^{t_j}[f'(t)-f'(\tau)]d\tau.$$

Hence for $t \in \Gamma_j$

(12) $$|f'(t)-e_j| \leq 2\lambda\omega(f', \|\Delta_N\|)$$

and

(13) $$|m_j - f'_j| \leq 14\lambda(2-\lambda)^{-1}\omega(f', \|\Delta_N\|).$$

Let $t=(t_{j-1}+t_j)/2+\varepsilon$. Then $|\varepsilon| \leq |h_j|/2+|\Gamma_j|/2 \leq (\lambda+1)|h_j|/2$. It follows from (7) $s'_{f,N}(t)-e_j=(3\varepsilon^2 h_j^{-2}-1/4)(m_{j-1}+m_j-2e_j)+\varepsilon(m_j-m_{j-1})/h_j$. Hence by (12) and (13) we obtain

$$|s'_{f,N}(t)-e_j| \leq (3(\lambda+1)^2+1)/4(|m_{j-1}-f'_{j-1}|+|f'_{j-1}-e_j|+|m_j-f'_j|+|f'_j-e_j|$$
$$+((\lambda+1)/2)(|m_j-f'_j|+|f'_j-f'_{j-1}|+|f'_{j-1}-m_{j-1}|) \leq A(\lambda)\omega(f', \|\Delta_N\|).$$

Hence $|s'_{f,N}(t)-f'(t)| \leq |s'_{f,N}(t)-e_j|+|e_j-f'(t)| \leq [A(\lambda)+2\lambda]\omega(f', \|\Delta_N\|)$ whence it follows that there exists a constant $A_{11}(\lambda)$ such that

(14) $$\|s'_{f,N}-f'\| \leq A_{11}(\lambda)\omega(f' \|\Delta_N\|).$$

Hence for $t \in \Gamma_j$ $|f(t)-s_{f,N}(t)| \leq \int_{t_{j-1}}^{t} |f'(\tau)-s'_{f,N}(\tau)| \, |d\tau| \leq \lambda \|\Delta_N\| A_{11}(\lambda)\omega(f', \|\Delta_N\|)$ whence it follows that there exists a constant $A_{10}(\lambda)$ such that

(15) $$\|f-s_{f,N}\| \leq A_{10}(\lambda)\|\Delta_N\| \, \omega(f', \|\Delta_N\|).$$

2) f is continuous on Γ. Let g be a spline of degree 1 with respect to the partition Δ_N of interpolation to the function f on Δ_N. Then $s_{f,N}=s_g$ and for $t \in \Gamma_j$ $g(t)=f_{j-1}(t_j-t)/h_j+f_j(t-t_{j-1})/h_j$, $g'(t)=(f_j-f_{j-1})/h_j$. Hence by (14)

$$|s_{f,N}(t)-g(t)| \leq \int_{t_{j-1}}^{t} |s'_{f,N}(\tau)-g'(\tau)| \, |d\tau| \leq 2A_{11}(\lambda)|h_j| \, \|g'\| \leq 2A_{11}(\lambda)K_N\omega(f, \|\Delta_N\|).$$

Because

(16) $$|f(t)-g(t)| \leq |f(t)-f_{j-1}| \, |t_j-t|/|h_j|+|f(t)-f_j| \, |t-t_{j-1}|/|h_j| \leq 2\lambda\omega(f, \|\Delta_N\|),$$

then there exists a constant $A_{00}(\lambda)$ such that

(17) $$\|f-s_{f,N}\| \leq A_{00}(\lambda)K_N\omega(f, \|\Delta_N\|).$$

3) f'' is continuous on Γ except at most finite number points at which it has one-sided limits (we set $f''(t)=f''(t+)$ at these points).

For $t \in \Gamma_j$ we have

(18) $$|f'(t)-(f_j-f_{j-1})/h_j| \leq |h_j|^{-1} \int_{t_{j-1}}^{t_j} |f'(t)-f'(\tau)| \, |d\tau|$$
$$\leq 2\lambda\omega(f', |h_j|) \leq 2\lambda^2 |h_j| \, \|f''\|.$$

Because

$$f[t_{j-1}, t_j, t_{j+1}]=(h_j+h_{j+1})^{-1}\{(h_{j+1}^{-1}(f_{j+1}-f_j)-f'_j)-(h_j^{-1}(f_j-f_{j-1})-f'_j)\}$$

and the quantities M_j satisfy the system (6), then $|M_j| \leq (2-\lambda)^{-1} 48\lambda^2 \|f''\|$, $j=1,\ldots,N$. For $t \in \Gamma_j$ we have $s''_{f,N}(t) = M_{j-1}(t_j - t)/h_j + M_j(t - t_{j-1})/h_j$. Hence

(19) $$\|s''_{f,N}\| \leq \frac{48\lambda^3}{2-\lambda} \|f''\|.$$

Further we need the following

Lemma. *If* $f \in C^2(\Gamma_1 \cup \Gamma_2)$ *and* g_f *is a quadratic spline with respect to the partition* $\Delta = \{t_0, t_1, t_2\}$ *satisfying the conditions*: $g_f^{(i)}(t_k) = f_k^{(i)}$ *for* $k=0, 2$, $i=0, 1$, *then there exist constants* $C_1(\lambda)$ *and* $C_2(\lambda)$ *such that*

(20) $$\|f' - g'_f\| \leq C_1(\lambda)\omega(f', \|\Delta\|)$$
$$\|f'' - g''_f\| \leq C_2(\lambda) K_N \omega(f'', \|\Delta\|).$$

Proof. We can write the function g_f as follows:

$$g_f(t) = f_0 + f'_0(t - t_0) + \tau_1(t - t_0)^2 + \tau_2(t - t_1)^2_+, \text{ where}$$

$$(t - t_1)^2_+ = \begin{cases} 0, & \text{for } t \in \Gamma_1, \\ (t - t_1)^2, & \text{for } t \in \Gamma_2. \end{cases}$$

By assumption the coefficients τ_1 and τ_2 satisfy the following system of equalities:

$$\tau_1(h_1 + h_2)^2 + \tau_2 h_2^2 = f_2 - f_1 - f'_0(h_1 + h_2),$$
$$2\tau_1(h_1 + h_2) + 2\tau_2 h_2 = f'_2 - f'_0.$$

Hence

$$\tau_1 = ((f_2 - f_0)/(h_1 + h_2) - f'_0)/h_1 - h_2(f'_2 - f'_0)/2h_1(h_1 + h_2),$$

whence by (18) we obtain

$$|\tau_1| \leq (\lambda^2 |h_1 + h_2| / |h_1| + \lambda |h_2| / 2|h_1|) \|f''\|, \quad |\tau_1| \leq |h_1|^{-1}(2\lambda + 1)\|f'\|.$$

Hence for $t \in \Gamma_1$

(21) $$|g''_f(t)| \leq \lambda K_N(2\lambda + 1)\|f''\|, \quad |g'_f(t)| \leq (4\lambda^2 + 2\lambda + 1)\|f'\|.$$

Now write the function g_f as follows:

$$g_f(t) = f_2 + f'_2(t - t_2) + \tau'_1(t - t_2)^2 + \tau'_2(t - t_1)^2_-, \quad \text{where}$$

$$(t - t_1)^2_- = \begin{cases} (t - t_1)^2, & \text{for } t \in \Gamma_1, \\ 0, & \text{for } t \in \Gamma_2. \end{cases}$$

Reasoning analogously as above we conclude that the inequalities (21) hold true also for $t \in \Gamma_2$. Let $s(t) = f''_1(t - t_2)^2/2$. Then $\|f'' - g''_f\| \leq \|f'' - s''\| + \|s'' - g''_f\| = \|f'' - f''_1\| + \|g''_{f-s}\|$. Hence by (21) we obtain the second inequality of (20).

Let now $s(t) = 2^{-1} f'_2(t - t_0)^2/(h_1 + h_2) - 2^{-1} f'_0(t_2 - t)^2/(h_1 + h_2)$. Because s' is a spline of degree 1 of interpolation to the function f' at the points t_0 and t_2, then by (16)

$$\|f'-s'\|\leq 4\lambda\omega(f', \|\Delta\|),$$

$$\|f'-g'_f\|\leq \|f'-s'\|+\|s'-g'_f\|=\|f'-s'\|+\|g'_{f-s}\|.$$

Hence by (21) we obtain the first inequality of (20).

Let $\Delta'_{2n}=\{t_0=t'_0, t'_1,\dots, t'_{2n}\}$ be a partition of the curve Γ such that each point of Δ'_{2n} is a point of Δ_N and $\|\Delta'_{2n}\|\leq 3\|\Delta_N\|$ with $K_{2n}\leq 6$. Let now g be a quadratic spline with respect to the partition Δ'_{2n} satisfying the following conditions:

(22) $$g^{(i)}(t'_{2j})=f^{(i+1)}(t'_{2j}), \quad i=0,\ 1, \quad j=1,\dots,\ n.$$

Set $s(t)=\int_{t_0}^{t} g(\tau)d\tau$. Hence by (19) and Lemma

$$\|f''-s''_{f,N}\|\leq\|f''-s''\|+\|s''-s''_{f,N}\|=\|f''-g'\|+\|s''_{f-s,N}\|$$

$$18C_1(\lambda)((2-\lambda)^{-1}48\lambda^3+1)\omega(f'', \|\Delta_N\|),$$

whence it follows that there exists a constant $A_{22}(\lambda)$ such that

(23) $$\|f''-s''_{f,N}\|\leq A_{22}(\lambda)\omega(f'', \|\Delta_N\|).$$

Because the function $s_{f,N}$ is interpolating to f on Δ_N and the function s' is satisfying the conditions (22), then by (14) and Lemma

$$\|f'-s'_{f,N}\|\leq \|f'-s'\|+\|s'-s'_{f,N}\|=\|f'-s'\|+\|s'_{f-s,N}\|\leq(2A_{11}(\lambda)+1)\|f'-s'\|$$

and

$$|f'(t)-s'(t)|\leq \int_{t'_{2j}}^{t} |f''(\tau)-s''(\tau)|\,|d\tau|\leq 3\lambda A_{22}(\lambda)\,\|\Delta_N\|\,\omega(f'', \|\Delta_N\|).$$

Hence there exist constants $A_{21}(\lambda)$ and $A_{20}(\lambda)$ such that

(24) $$\|f'-s'_{f,N}\|\leq A_{21}(\lambda)\,\|\Delta_N\|\omega(f'',\|\Delta_N\|),\quad \|f-s_{f,N}\|\leq A_{20}(\lambda)\,\|\Delta_N\|^2\omega(f'',\|\Delta_N\|).$$

4) f''' is continuous on Γ except at most finite number points at which it has one-sided limits.

For $t\in\Gamma_j$ we have $s'''_{f,N}(t)=(M_j-M_{j-1})/h_j$.

By (23)

$$|s'''_{f,N}(t)|\leq|h_j|^{-1}(|M_j-f''_j|+|f''_j-f''_{j-1}|+|f''_{j-1}-M_{j-1}|)$$

$$\leq|h_j|^{-1}(2A_{22}(\lambda)+1)\,\omega(f'',\|\Delta_N\|)\leq\lambda K_N(2A_{22}(\lambda)+1)\|f''\|.$$

Applying the Lemma we obtain

$$\|f'''-s'''_{f,N}\|\leq\|f'''-s'''\|+\|s'''-s'''_{f,N}\|=\|f'''-g''\|+\|s'''_{f-s,N}\|$$

$$\leq 18C_2(\lambda)\{\lambda K_N[2A_{22}(\lambda)+1]+1\}\omega(f''', \|\Delta_N\|\}.$$

Hence there exists a constant $A_{33}(\lambda)$ such that

(25) $$\|f'''-s'''_{f,N}\|\leq A_{33}(\lambda)K_N\omega(f''', \|\Delta_N\|).$$

We obtain the remaining inequalities of (10) analogously.

3. **Approximation by Analytic Splines.** Under the above assumption on the curve Γ and the partition Δ_N we shall prove the following

Theorem 2. *If the function f is analytic in D and of class C^p in $\bar{D}=D\cup\Gamma$, $p=0, 1, 2, 3$, then there exist constants $B_{l,k}(\Gamma)$ depending only on Γ such that*

$$\| f-S_{f,N}\|\leq\begin{cases} B_{p0}(\Gamma)K_N\|\Delta_N\|^p\ln\dfrac{1}{\|\Delta_N\|}\,\omega(f^{(p)},\ \|\Delta_N\|), & p=0,\ 3,\\[2ex] B_{p0}(\Gamma)\|\Delta_N\|^p\ln\dfrac{1}{\|\Delta_N\|}\,\omega(f^{(p)},\ \|\Delta_N\|), & p=1,\ 2,\end{cases}$$

$$\|f'-S'_{f,N}\|\leq B_{21}(\Gamma)\|\Delta_N\|\omega(f'',\ \|\Delta_N\|),\quad p=2, \tag{26}$$

$$\|f^{(i)}-S^{(i)}_{f,N}\|\leq B_{pi}(\Gamma)K_N\|\Delta_N\|^{p-i}\omega(f^{(p)},\ \|\Delta_N\|),\quad i=1,\ 2,\ p=1, 2, 3,\ i\leq p.$$

Remark. Let $\omega_\Gamma(f, h)=\sup\{|f(t_2)-f(t_1)|: t_1,\ t_2\in\Gamma,\ |t_2-t_1|\leq h\}$. P. M Tamrazov proved in [7] that

$$\omega(f, h)\leq 108\omega_\Gamma(f, h). \tag{27}$$

Hence we can replace the modulus of continuity $\omega(f, h)$ by the modulus of continuity $\omega_\Gamma(f, h)$ in Theorem 2.

Corollary. *If the function f is analytic in D, $f\in C^p(D\cup\Gamma)$, $p=0$, 1, 2, 3 and the function $f^{(p)}$ satisfies a Hölder condition of order β on Γ, $0<\beta\leq 1$ and $K_N\leq K<\infty$, then for any β', $0<\beta'<\beta$ we have*

$$\|f^{(i)}-S^{(i)}_{f,N}\|=o(\|\Delta_N\|^{p-i+\beta'}),\quad i=0,\ 1,\ 2,\quad i\leq p, \tag{28}$$

for $i=1$, 2, $\beta'=\beta$ and for $i=1$, $p=2$ we have no restriction on the values K_N.

Proof of Theorem 2. By the principle of maximum it is sufficient to investigate the convergence of $\{S_{f,N}(z)\}$ on Γ.

1) $f\in C(D\cup\Gamma)$. Let $t\in\Gamma$. Because the addition of a constant to the function f changes neither the modulus of continuity of f nor the difference $f(z)-S_{f,N}(z)$ then we can assume that $f(t)=0$.

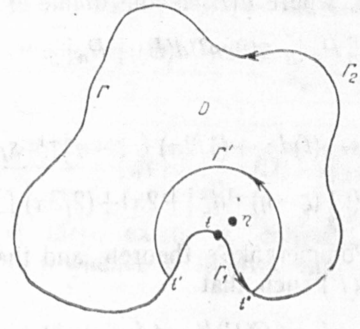

Fig. 1

Let $\varepsilon>0$, $\eta\in D$, $|t-\eta|<\|\Delta_N\|/4$ and $|f(\eta)-f(t)|<\varepsilon$ (Fig. 1). Let points t' and t'' divide the curve Γ on two arcs Γ_1 and Γ_2 as follows: t' and t'' are the first and the second points (with respect to the orientation of Γ) of the set $\Gamma\cap\{z: |z-t|=\|\Delta_N\|\}$, Γ_1 is the arc from t' to t'', Γ' is the part of the boundary of the region $D\cap\{z:|z-t|<\|\Delta_N\|\}$ inside D, $|\Gamma|=$ the length of Γ.

$$f(\eta)-S_{f,N}(\eta)=(1/2\pi i)\int_{\Gamma}(\zeta-\eta)^{-1}(f(\zeta)-s_{f,N}(\zeta))d\zeta=(1/2\pi i)\int_{\Gamma_2}(\zeta-\eta)^{-1}(f(\zeta)-s_{f,N}(\zeta))d\zeta$$
$$+(1/2\pi i)\int_{\Gamma_1}(\zeta-\eta)^{-1}f(\zeta)d\zeta+(1/2\pi i)\int_{\Gamma_1}(\zeta-\eta)^{-1}s_{f,N}(\zeta)d\zeta=I_1+I_2+I_3.$$

Hence $|I_1|\leq(1/2\pi)\|f-s_{f,N}\|\int_{\Gamma_2}|\zeta-\eta|^{-1}|d\zeta|$. Let $x\in[0,|\Gamma|]$ be the natural parameter of the curve Γ such that $\zeta(0)=t$. Because the curve Γ is of class $S_{h,\lambda}$ then $x\leq\lambda|\zeta(x)-t|$. Because of $|\zeta'|\leq 1$ we have

$$\int_{\Gamma_2}\frac{|d\zeta|}{|\zeta-\eta|}\leq 2\lambda\int_{\|\Delta_N\|}^{h}|\zeta'|\left|\frac{\zeta(x)-t}{\zeta(x)-\eta}\right|\frac{dx}{x}+2\lambda\int_{h}^{|\Gamma|/2}|\zeta'|\left|\frac{\zeta(x)-t}{\zeta(x)-\eta}\right|\frac{dx}{x}\leq\frac{8}{3}\int_{\|\Delta_N\|}^{h}\frac{dx}{x}+\frac{4}{3h}|\Gamma|.$$

Hence there exists a constant $A(\Gamma)$ such that

$$|I_1|\leq A(\Gamma)\|f-s_{f,N}\|\ln\frac{1}{\|\Delta_N\|}. \tag{29}$$

By the Cauchy integral formula we obtain $I_2=\int_{\Gamma_1}(\zeta-\eta)^{-1}f(\zeta)d\zeta=f(\eta)-\int_{\Gamma'}(\zeta-\eta)^{-1}f(\zeta)d\zeta$. Because Γ' is a curve of class $S_{h,\lambda}(\lambda<2)$, then there exists a constant $B(\lambda)$ such that $\int_{\Gamma'}|(\zeta-\eta)^{-1}|f(\zeta)||d\zeta|\leq B(\lambda)\|f\|_{\Gamma}$.

Hence

$$|I_2|\leq B(\Gamma)\omega(f,\|\Delta_N\|)+\varepsilon. \tag{30}$$

Further we need the following

Theorem (Ch. Pommerenke cf. [4, 6]). *Let B be a closed limited point set containing more than one point such that, $\mathbb{C}\setminus B$ is connected and let $\gamma(B)$ be the analytic capacity of the set B. Then for any polynomial P_n of degree n*

$$\|P_n'\|_B\leq(en^2/2\gamma(B))\|P_n\|_B,\ \textit{where}\ \|f\|_B=\sup\{|f(z)|,\ z\in B\}. \tag{31}$$

Because $\gamma(B)\leq d(B)/4$, where $d(B)$ is the diameter of B, then

$$\|P_n'\|_B\leq(2en^2/d(B))\|P_n\|_B. \tag{32}$$

Further

$$|I_3|\leq(1/2\pi)\left|\int_{\Gamma_1}(\zeta-\eta)^{-1}s_{f,N}(t)d\zeta\right|+(1/2\pi)\int_{\Gamma_1}|\zeta-\eta|^{-1}|s_{f,N}(\zeta)-s_{f,N}(t)||d\zeta|$$
$$\leq(|s_{f,N}(t)|/2\pi)\left(\left|\int_{\Gamma'}(\zeta-\eta)^{-1}d\zeta\right|+2\pi\right)+(2/3\pi)\int_{\Gamma_1}\|s_{f,N}'\|\,|d\zeta|.$$

Because of $f(t)=0$, Pommerenke's theorem and the proof of Theorem 1 there exists a constant $C(\Gamma)$ such that

$$|I_3|\leq C(\Gamma)K_N\omega(f,\|\Delta_N\|). \tag{33}$$

Adding (29), (30) and (33), tending with ε to zero and because of Theorem 1 we conclude that there exists a constant $B_{00}(\Gamma)$ such that

$$\|f-S_{f,N}\|\leq B_{00}(\Gamma)K_N\ln\frac{1}{\|\Delta_N\|}\omega(f,\|\Delta_N\|). \tag{34}$$

2) $f \in C^1(D \cup \Gamma)$. Because the addition of a linear function to f changes neither the modulus of continuity of f nor the differences $f(z)-S_{f,N}(z)$ and $f'(z)-S'_{f,N}(z)$, then we can assume that $f(t)=f'(t)=0$. Let $\varepsilon>0$, $\eta \in D$, $|t-\eta| \leq \|\Delta_N\|/4$, $|f(\eta)-f(t)|<\varepsilon$ and $|f'(\eta)-f'(t)|<\varepsilon$. We have

$$f'(\eta)-S'_{f,N}(\eta)=(1/2\pi i)\int_{\Gamma}(\zeta-\eta)^{-1}(f'(\zeta)-s'_{f,N}(\zeta))d\zeta$$
$$=(1/2\pi i)\int_{\Gamma_2}(\zeta-\eta)^{-1}(f'(\zeta)-s'_{f,N}(\zeta))d\zeta$$
$$+(1/2\pi i)\int_{\Gamma_1}(\zeta-\eta)^{-1}f'(\zeta)d\zeta+(1/2\pi i)\int_{\Gamma_1}(\zeta-\eta)^{-1}s'_{f,N}(\zeta)d\zeta=I_1+I_2+I_3.$$

Further

$$2\pi i I_1=(\zeta-\eta)^{-1}(f(\zeta)-s_{f,N}(\zeta)\Big|_{t''}^{t'}-\int_{\Gamma_2}(\zeta-\eta)^{-2}(f(\zeta)-s_{f,N}(\zeta))d\zeta.$$

Hence by Theorem 1

$$|I_1|\leq A_{10}(\lambda)\omega(f',\|\Delta_N\|)((4/3\pi)+2\|\Delta_N\|\int_{\|\Delta_N\|}^{h}\|\zeta'\|\left|\frac{\zeta-t}{\zeta-\eta}\right|^2\frac{dx}{x^2}$$
$$+2\|\Delta_N\|\int_{h}^{|\Gamma|/2}\|\zeta'\|\left|\frac{\zeta-t}{\zeta-\eta}\right|^2\frac{dx}{x^2}).$$

Hence there exists a constant $A'(\Gamma)$ such that $|I_1|\leq A'(I)\omega(f',\|\Delta_N\|)$. We estimate the integrals I_2 and I_3 analogously as in 1). So we prove that there exists a constant $B_{11}(\Gamma)$ such that

$$\|f'-S'_{f,N}\|\leq B_{11}(\Gamma)K_N\omega(f',\|\Delta_N\|), \tag{35}$$

Further

$$f(\eta)-S_{f,N}(\eta)=(1/2\pi i)\int_{\Gamma_2}(\zeta-\eta)^{-1}(f(\zeta)-s_{f,N}(\zeta))d\zeta$$
$$+(1/2\pi i)\int_{\Gamma_1}(\zeta-\eta)^{-1}(f(\zeta)-s_{f,N}(\zeta))d\zeta=I_1+I_2.$$

We estimate the integral I_1 analogously as the integral I_1 in 1 and

$$|I_2|\leq\frac{1}{2\pi}\int_{\Gamma_1}\frac{|[f(\zeta)-s_{f,N}(\zeta)]-[f(t)-s_{f,N}(t)]|}{|\zeta-\eta|}|d\zeta|+\frac{1}{2\pi}\left|\int_{\Gamma_1}\frac{f(t)-s_{f,N}(t)}{\zeta-\eta}d\zeta\right|$$

$$\leq\frac{1}{2\pi}\int_{\Gamma_1}\frac{\|f'-s'_{f,N}\|\,|\zeta-t|}{|\zeta-\eta|}|d\zeta|+\frac{1}{2\pi}|f(t)-s_{f,N}(t)|\left(\left|\int_{\Gamma'}\frac{d\zeta}{\zeta-\eta}\right|+2\pi\right).$$

Hence by Theorem 1 there exists a constant $B'(\Gamma)$ such that $|I_2| \leq B'(\Gamma)\|\Delta_N\|\omega(f',\|\Delta_N\|)$ whence it follows that there exists a constant $B_{10}(\Gamma)$ such that

$$\|f-S_{f,N}\|\leq B_{10}(\Gamma)\|\Delta_N\|\ln\frac{1}{\|\Delta_N\|}\omega(f',\|\Delta_N\|). \tag{36}$$

We obtain the remaining inequalities of (26) analogously.

REFERENCES

1. J. H. Ahlberg, E. N. Nilson, J. L. Walsh. Complex Cubic Splines. *Trans. Amer. Math. Soc.*, **129**, 1967, 391—413.
2. А. О. Гельфонд. Исчисление конечных разностей. Москва, 1967.
3. F. Leja. Teoria funkcji analytycznych. Warszawa, 1957.
4. Ch. Pommerenke. On the derivative of a polynomial. *Michigan Math. J.*, 1959, 6 (4), 373—375.
5. И. И. Привалов. Граничные свойства аналитических функций. Москва, 1950.
6. В. И. Смирнов, Н. А. Лебедев. Конструктивная теория функций комплексного переменного. Москва, 1964.
7. П. М. Тамразов. Гладкости и полиномиальные приближения. Киев, 1975.

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