КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'81
София, 1983, с. 205—210

CONSTRUCTIVE FUNCTION THEORY'81
Sofia, 1983, p. 205—210

A GENERALIZATION OF A SEQUENCE OF LINEAR POSITIVE OPERATORS

J. A. H. Alkemade

Summary. In this paper the sequence of linear positive operators $\{A_n\}_1^\infty$, defined on $M[0,\infty)$ (see Section 1) by

$$(A_n f)(x)=(1-\tau_n(x))^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \tau_n^k(x) f((k+c_1k^\alpha)(k+n+c_2k^\beta)/\rho_n), \quad x\in[0,\infty),$$

is investigated. Here $\{\tau_n(x)\}_1^\infty$ is the sequence of functions, given in (1.2), and $\{\rho_n\}_1^\infty$ is a sequence of positive numbers with the property (1.1). We shall prove that under certain conditions concerning α and β the sequence $\{(A_n f)(x)\}_1^\infty$, where f is continuous in the point x, tends to $f(x)$ as $n\to\infty$. Furthermore a Voronovskaya-type formula is derived for the difference $(A_n f)(x)-f(x)$ where $f\in H^{(2)}(x)$ (see Section 1 for the definition of this class of functions), which gives the speed of approximation.

1. Introduction. Let $M[0,\infty)$ be the set of functions $f(t)$, defined on $[0,\infty)$, for which there exist two constants $A\geqq 0$ and $B\geqq 0$ and an $m\in N$, such that $|f(t)|\leqq A+Bt^m$, $t\in[0,\infty)$. Here A, B and m may depend on f. We choose the sequence of positive numbers $\{\rho_n\}_1^\infty$ such that

$$\lim_{n\to\infty} n/\rho_n=0. \tag{1.1}$$

The sequence of functions $\{\tau_n(x)\}_1^\infty$ is defined on $[0,\infty)$ by

$$\tau_n(x)=2\rho_n x/(2\rho_n x+n(n+1)+(4\rho_n n(n+1)x+n^2(n+1)^2))^{1/2}. \tag{1.2}$$

Now the sequence of operators $\{\mathscr{A}_n\}_1^\infty$ is defined on $M[0,\infty)$ by

$$(\mathscr{A}_n f)(x)=(1-\tau_n(x))^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \tau_n^k(x) f(k(k+n)/\rho_n), \quad x\in[0,\infty).$$

Obviously $\mathscr{A}_n$ is linear and positive. In [1] it has been proved that $\mathscr{A}_n f\in C[0,\infty)$. Here $C[0,\infty)$ is the class of functions, which are continuous in each point $x\in[0,\infty)$. Furthermore it has been proved that, if $f\in C[0,\infty)$ and f bounded on $[0,\infty)$, $\lim_{n\to\infty}(\mathscr{A}_n f)(x)=f(x)$, pointwise in x. Also a Voronovskaya-type formula has been derived.

The operator $\mathscr{A}_n$ is now generalized as follows. Let $c_1 \geq 0$, $c_2 \geq -1$, $\alpha \geq 0$ and $\beta \geq 0$. Then the sequence of linear positive operators $\{A_n\}_1^\infty$ is defined on $M[0, \infty)$ by

$$(1.3) \quad (A_n f)(x) = (1-\tau_n(x))^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \tau_n^k(x) f((k+c_1k^\alpha)(k+n+c_2k^\beta)/\rho_n),$$

$x \in [0, \infty)$. It is easily proved that $A_n f \in C[0, \infty)$ for each $f \in M[0, \infty)$. In this paper it is investigated under which conditions concerning α and β the sequence $\{(A_n f)(x)\}_1^\infty$, where f is continuous in x, tends to $f(x)$ as $n \to \infty$. Also the speed of this approximation is considered by deriving a Voronovskaya-type formula for the difference $(A_n f)(x) - f(x)$, where $f \in H^{(2)}(x)$. Here $H^{(2)}(x)$ is the set of functions, defined on $[0, \infty)$, for which $f''(x)$ exists, while $f(t) = O(t^2)$ as $t \to \infty$ and f is bounded on each interval $[0, b]$ $(b > 0)$. Obviously $H^{(2)}(x) \subset M[0, \infty)$.

We have considered three cases concerning the speed with which ρ_n tends to infinity as $n \to \infty$, separately, namely $\rho_n/n^2 \to 0$, $\rho_n/n^2 \to a$ $(a > 0)$, and $\rho_n/n^2 \to \infty$, as $n \to \infty$. The first case is treated in extension in Sections 2, 3 and 4, while in Section 5 the results for the other two cases are given.

2. Preliminaries. In Sections 2, 3 and 4 we shall study the case in which

$$(2.1) \qquad \lim_{n \to \infty} \rho_n/n^2 = 0,$$

while property (1.1) also holds. In this section the asymptotic behaviour of $S_{n\gamma}(\tau_n(x))$, $(\gamma \geq 0)$, and $T_{nm}(\tau_n(x))$ $(m = 1, 2, 3, 4)$, as $n \to \infty$ is investigated. Here the function $S_{n\gamma}(y)$ $(n = 1, 2, \ldots; \gamma \geq 0)$ is defined by

$$(2.2) \qquad S_{n\gamma}(y) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} k^\gamma y^k, \quad y \in [0, 1),$$

and $T_{nm}(y)$ $(n = 1, 2, \ldots; m = 0, 1, \ldots)$, by

$$T_{nm}(y) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} k^m (k+n)^m y^k, \quad y \in [0, 1).$$

Lemma. *Let $\{\tau_n(x)\}_1^\infty$ be the sequence of functions defined by* (1.2), *while $\{\rho_n\}_1^\infty$ satisfies* (1.1) *and* (2.1). *Then for $x \in [0, \infty)$, x fixed and $\gamma \geq 0$*

$$(1-\tau_n(x))^n S_{n\gamma}(\tau_n(x)) = (\rho_n/n)^\gamma x^\gamma + o((\rho_n/n)^\gamma) \quad as \quad n \to \infty.$$

Proof. It is easy to see from (1.2) and (2.1) that for $x \in [0, \infty)$, x fixed

$$(2.3) \qquad \tau_n(x) = \rho_n x/n^2 + o(\rho_n/n^2) \quad \text{as} \quad n \to \infty,$$

while in particular $\tau_n(0) = 0$ for each n. From (2.2) it directly follows that $S_{n0}(y) = 1/(1-y)^n$. Further it is possible for $m = 1, 2, \ldots$ to express $S_{nm}(y)$ in terms of functions $S_{n+1,i}(y)$ $(i = 0, 1, \ldots, m-1)$. In fact

$$S_{nm}(y) = ny \sum_{i=0}^{m-1} \binom{m-1}{i} S_{n+1,i}(y).$$

With the aid of this formula it is possible to prove by induction that

$$(2.4)\qquad S_{nm}(y)=(1-y)^{-n}\sum_{k=0}^{m-1} c_{km}(n)_{m-k}(y/(1-y))^{m-k},\quad y\in[0,1),$$

for $m=1,2,\ldots$, where $c_{0m}=1$, $c_{1m}=\binom{m}{2}$ and $c_{m-1,m}=1$ and $(n)_k=n(n+1)\ldots(n+k-1)$, $k\geqq 1$. For a complete proof of (2.4) see [2, pages 11-14]. With (2.3) it follows that $n^m(\tau_n(x)/(1-\tau_n(x)))^m=(\rho_n/n)^m(x^m+o(1))$, as $n\to\infty$, for $m=0,1,\ldots$ and $x\in[0,\infty)$, x fixed. Because $\tau_n:[0,\infty)\to[0,1)$, $S_{n\gamma}(\tau_n(x))$ is well defined and from (2.4) and (1.1) it follows that for $m=1,2,\ldots$ and x fixed

$$(2.5)\qquad (1-\tau_n(x))^n S_{nm}(\tau_n(x))=(\rho_n/n)^m(x^m+o(1))\quad\text{as } n\to\infty,$$

while for $m=0$ this is also true. So the lemma is proved for $\gamma=0,1,\ldots$ Let us now define the operator D_n on $M[0,\infty)$ by

$$(D_nf)(x)=(1-\tau_n(x))^n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\tau_n^k(x)f(nk/\rho_n),\quad x\in[0,\infty).$$

Let e_l $(l=0,1,\ldots)$ be the function defined by $e_l:t\to t^l$, $t\in[0,\infty)$. Then it follows from (2.2) and (2.5) that for each $x\in[0,\infty)$

$$(2.6)\qquad \begin{aligned}&(D_ne_0)(x)=1\ \text{for each } n,\\ &(D_ne_l)(x)=x^l+o(1)\ \text{as } n\to\infty\ \text{for } l=1,2,\ldots.\end{aligned}$$

Now let $f\in M[0,\infty)$ and f continuous in x. Let $\varepsilon>0$. There exists a $\delta>0$ such that $|f(t)-f(x)|<\varepsilon/2$ if $|t-x|<\delta$. Because $f\in M[0,\infty)$ there exists a $b>0$ and an $m\in N$ such that $|f(t)-f(x)|\leqq b(t-x)^{2m}$ if $|t-x|\geqq\delta$. Thus we have with (2.6) and the positivity and linearity of D_n

$$|(D_nf)(x)-f(x)|\leqq(1-\tau_n(x))^n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\tau_n^k(x)\,|f(nk/\rho_n)-f(x)|<\frac{1}{2}\varepsilon$$
$$+b(D_n(t-x)^{2m})(x).$$

From (2.6) it follows that $(D_n(t-x)^{2m})(x)=o(1)$ as $n\to\infty$, thus if n is large enough, then $|(D_nf)(x)-f(x)|<\varepsilon$, so

$$(2.7)\qquad \lim_{n\to\infty}(D_nf)(x)=f(x).$$

Taking $f(t)=t^\gamma$ $(\gamma\geqq 0)$ in (2.7) we arrive at

$$(n/\rho_n)^\gamma(1-\tau_n(x))^nS_{n\gamma}(\tau_n(x))=x^\gamma+o(1)\quad\text{as } n\to\infty,$$

and the lemma is proved.

Remark. In [2, p. 66-73] this lemma has been proved in a different way and only for a special choice of $\{\rho_n\}_1^\infty$ and $\{\tau_n(x)\}_1^\infty$. The proof, showed here, is more convenient and general.

The function $T_{nm}(y)$ can be written for $m=1,2,\ldots$ in terms of $T_{n+2,i}(y)$ $(i=0,1,\ldots,m-1)$. In fact

$$T_{nm}(y)=n(n+1)y\sum_{i=0}^{m-1}\binom{m-1}{i}(n+1)^{m-i-1}T_{n+2,i}(y).$$

Because $T_{n0}(y)=1/(1-y)^n$ it is possible to determine $T_{nm}(y)$. From (1.2) we can derive that $\tau_n(x)/(1-\tau_n(x))^2=\rho_n x/n(n+1)$. With this property it is possible to derive the next formulas by straightforward calculation:

$$(2.8)\quad \begin{aligned}
&(1-\tau_n(x))^n T_{n1}(\tau_n(x))=\rho_n x,\\
&(1-\tau_n(x))^n T_{n2}(\tau_n(x))=\rho_n^2 x^2+n\rho_n x+o(n\rho_n),\\
&(1-\tau_n(x))^n T_{n3}(\tau_n(x))=\rho_n^3 x^3+3n\rho_n^2 x^2+o(n\rho_n^2),\\
&(1-\tau_n(x))^n T_{n4}(\tau_n(x))=\rho_n^4 x^4+6n\rho_n^3 x^3+o(n\rho_n^3), \quad \text{as} \quad n\to\infty.
\end{aligned}$$

3. An Approximation Theorem. With the formulas, derived in Section 2, it is possible to determine the asymptotic behaviour of $(A_n e_l)(x)$ as $n\to\infty$, x fixed.

Theorem 1. *Let $\{A_n\}_1^\infty$ be the sequence of operators, defined in* (1.3), *while* (2.1) *holds. Let $x\in[0,\infty)$, x fixed. If $0\leq\alpha<1$ and $\beta\geq 0$ such that $(\rho_n/n)^\beta=o(n)$ as $n\to\infty$, then*

(i) $$(A_n e_l)(x)=x^l+o(1)\quad (l=0,1,\ldots),$$

and particularly, setting $\varphi(n)=(n/\rho_n)^{1-\alpha}+(\rho_n/n)^\beta n^{-1}$,

(ii) $$(A_n e_0)(x)=1,\quad x\in[0,\infty),$$

(iii) $$(A_n e_1)(x)=x+c_1(n/\rho_n)^{1-\alpha}x^\alpha+c_2(\rho_n/n)^\beta n^{-1}x^{1+\beta}+o(\varphi(n)),$$

(iv) $$(A_n e_2)(x)=x^2+nx/\rho_n+2c_1(n/\rho_n)^{1-\alpha}x^{1+\alpha}+2c_2(\rho_n/n)^\beta n^{-1}x^{2+\beta}+o(\varphi(n)),$$

(v) $$(A_n e_3)(x)=x^3+3nx^2/\rho_n+3c_1(n/\rho_n)^{1-\alpha}x^{2+\alpha}+3c_2(\rho_n/n)^\beta n^{-1}x^{3+\beta}+o(\varphi(n)),$$

(vi) $$(A_n e_4)(x)=x^4+6nx^3/\rho_n+4c_1(n/\rho_n)^{1-\alpha}x^{3+\alpha}+4c_2(\rho_n/n)^\beta n^{-1}x^{4+\beta}+o(\varphi(n)),$$

as $n\to\infty$.

Proof. (i). For $n=1,2,\ldots$, $l=0,1,\ldots$ and $x\in[0,\infty)$

$$(A_n e_l)(x)=(1-\tau_n(x))^n\rho_n^{-l}\sum_{i=0}^{l}\sum_{j=0}^{l}\binom{l}{i}\binom{l}{j}c_1^i c_2^j\sum_{m=0}^{l-j}n^m\binom{l-j}{m}S_{n,2l-i-j+i\alpha+j\beta-m}(\tau_n(x)).$$

Let first $x>0$, x fixed. With the lemma and the fact that $0\leq\alpha<1$ it follows that only the terms for which $i=0$ can contribute to the leading term of $(A_n e_l)(x)$. Moreover $S_{n,\gamma+1}(\tau_n(x))=o(nS_{n\gamma}(\tau_n(x)))$ as $n\to\infty$, thus we have to consider only the terms for which $m=l-j$ to determine the leading term. Thus with the lemma

$$(A_n e_l)(x)=\rho_n^{-l}(1-\tau_n(x))^n\sum_{j=0}^{l}\binom{l}{j}c_2^j n^{l-j}S_{n,l+j\beta}(\tau_n(x))(1+o(1))$$

$$=\sum_{j=0}^{l}\binom{l}{j}c_2^j\rho_n^{j\beta}x^{l+j\beta}/n^{j(\beta+1)}(1+o(1)),$$

as $n\to\infty$. For $j\geq 1$ we have $\rho_n^{j\beta}/n^{j(\beta+1)}=o(1)$ as $n\to\infty$ and thus (i) has been proved for $x>0$. Let now $x=0$. Then $\tau_n(0)=0$ and for $l=1,2,\ldots$

$$(3.1)\quad (A_n e_l)(0)=(c_1 0^\alpha(n+c_2 0^\beta)/\rho_n)^l=o(1),\quad \text{as } n\to\infty,$$

because of (1.1). For $l=0$ we have $(A_n e_0)(0)=1$ and thus (i) has been proved for $x=0$.

(ii) is evident, because $S_{n0}(y)=1/(1-y)^n$.

(iii) If $x=0$, the third relation is clear from (3.1). So let now $x>0$. Because $(k+c_1k^\alpha)(k+n+c_2k^\beta)=k(k+n)+c_1k^{1+\alpha}+c_1nk^\alpha+c_2k^{1+\beta}+c_1c_2k^{\alpha+\beta}$,

$$(A_ne_1)(x)=\rho_n^{-1}(1-\tau_n(x))^n\{T_{n1}(\tau_n(x))+c_1S_{n,\alpha+1}(\tau_n(x))+c_1nS_{n\alpha}(\tau_n(x))$$
$$+c_2S_{n,\beta+1}(\tau_n(x))+c_1c_2S_{n,\alpha+\beta}(\tau_n(x))\}.$$

With (2.8) and the lemma it follows that

$$(A_ne_1)(x)=x+c_1(n/\rho_n)^{1-\alpha}x^\alpha+c_2(\rho_n/n)^\beta n^{-1}x^{1+\beta}+o(\varphi(n)),$$

as $n\to\infty$. Thus (iii) is proved.

In the same way it is possible to prove the parts (iv) till (vi). In analogy with the proof (2.7) the next approximation theorem concerning the sequence of operators $\{A_n\}_1^\infty$ can be proved.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. For $f\in M[0,\infty)$ and f continuous in x, the relation $\lim_{n\to\infty}(A_nf)(x)=f(x)$ holds.*

4. The Speed of Approximation. It is just shown that under certain conditions concerning α and β the sequence $\{(A_nf)(x)\}_1^\infty$ tends to $f(x)$ as $n\to\infty$, where f is continuous in x. In this section the speed, with which this approximation takes place, is investigated by deriving a Voronovskaya-type formula.

Theorem 3. *Let the conditions of Theorem 1 be fulfilled. For $f\in H^{(2)}(x)$*

$$(A_nf)(x)-f(x)=\{c_1(n/\rho_n)^{1-\alpha}x^\alpha+c_2(\rho_n/n)^\beta n^{-1}x^{1+\beta}\}f'(x)+nxf''(x)/2\rho_n+o(\varphi(n)),$$

as $n\to\infty$. Here $\varphi(n)=(n/\rho_n)^{1-\alpha}+(\rho_n/n)^\beta n^{-1}$ as in Theorem 1.

Proof. Because $f\in H^{(2)}(x)$ we can write for $t\in[0,\infty)$

$$f(t)=f(x)+(t-x)f'(x)+(t-x)^2f''(x)/2+(t-x)^2h(t-x), \tag{4.1}$$

where $h(u)$ is bounded on $[-x,\infty)$. If $h(0)=0$ is taken, the function $h(u)$ is continuous in $u=0$. Applying A_n to both sides of (4.1), we have because of the linearity of A_n and (ii), (iii) and (iv) of Theorem 1

$$(A_nf)(x)-f(x)=\{c_1(n/\rho_n)^{1-\alpha}x^\alpha+c_2(\rho_n/n)^\beta n^{-1}x^{1+\beta}\}f'(x)$$
$$+nxf''(x)/2\rho_n+(A_n(t-x)^2h(t-x))(x)+o(\varphi(n)),$$

as $n\to\infty$. Let $\varepsilon>0$, arbitrary chosen. Because $h(u)$ is continuous in $u=0$ there exists a $\delta>0$, such that $u^2|h(u)|<\varepsilon$ if $|u|<\delta$. Furthermore $h(u)$ is bounded on $[-x,\infty)$, say $|h(u)|\le M$, $(M>0)$, so if $|u|\ge\delta$, $u^2|h(u)|\le(M/\delta^4)u^4=bu^4$. And thus $(t-x)^2|h(t-x)|<\varepsilon+b(t-x)^4$ and because of the positivity and linearity of A_n it follows with Theorem 1 that

$$|(A_n(t-x)^2h(t-x))(x)|\le(A_n(t-x)^2|h(t-x)|)(x)$$
$$<\varepsilon+b(A_n(t-x)^4)(x)=\varepsilon+o(\varphi(n))\quad\text{as } n\to\infty.$$

Because $\varepsilon>0$ can be chosen arbitrary small we arrive at the assertion in Theorem 3.

5. The Other Two Cases. In this section only the results are given for the cases that $\rho_n/n^2\to a$ $(a>0)$, and $\rho_n/n^2\to\infty$ as $n\to\infty$, respectively.

Theorem 4. *Assume* $\rho_n/n^2 \to a$ $(a>0)$, *or* $\rho_n/n^2 \to \infty$ *as* $n\to\infty$. *Let* $f \in M[0, \infty)$, *where* f *is continuous in* x, x *fixed. If* $0 \leq \alpha < 1$ *and* $0 \leq \beta < 1$, *then* $\lim_{n\to\infty} (A_n f)(x) = f(x)$.

The formulas for the difference $(A_n f)(x) - f(x)$ are different in both cases. We first consider the case that $\rho_n/n^2 \to a$ $(a>0)$. Let $\tau(x) = 2x/(2x+1 + \sqrt{(4x+1)})$ for $x \in [0, \infty)$. If $0 \leq \alpha < 1$, $0 \leq \beta < 1$ and $f \in H^{(2)}(x)$, the next formula holds:

$$(A_n f)(x) - f(x) = \{c_1 a^{-1}(\tau(ax)/(1-\tau(ax)))^{\alpha}(1-\tau(ax))^{-1} n^{\alpha-1}$$

$$+ c_2 a^{-1}(\tau(ax)/(1-\tau(ax)))^{\beta+1} n^{\beta-1}\} f'(x) + n^{-1}(4x^2 + x/a) f''(x)/2 + o(n^{\alpha-1} + n^{\beta-1}),$$

as $n\to\infty$, $x \in [0, \infty)$, x fixed.

Let now $\rho_n/n^2 \to \infty$ as $n\to\infty$. If $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, then for $x \in [0, \infty)$, x fixed $(A_n f)(x) - f(x) = \{c_1 \rho_n^{(\alpha-1)/2} x^{(\alpha+1)/2} + c_2 \rho_n^{(\beta-1)/2} x^{(\beta+1)/2}\} f'(x) + 2n^{-1} x^2 f''(x) + o(n^{-1} + \rho_n^{(\alpha-1)/2} + \rho_n^{(\beta-1)/2})$, as $n\to\infty$, where $f \in H^{(2)}(x)$.

REFERENCES

1. H. J. Timmermans. Een nieuw type positieve lineaire operatoren in de approximatietheorie. Report of the Department of Mathematics, University of Technology, Delft, 1979, 39 p.
2. J. A. H. Alkemade. Generalisatie van een rij van lineaire positieve operatoren van het sommatietype. Report of the Department of Mathematics, University of Technology, Delft, 1981, 120 p.

University of Technology
Department of Mathematics
Delft The Netherlands

Received on July 15, 1981