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ON GLOBAL APPROXIMATION BY KANTOROVITCH POLYNOMIALS IN L^p

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Summary. The purpose of this note is to extend our previous results on global approximation by Kantorovitch polynomials in L^1 to the spaces L^p, $1<p<\infty$. This includes some further characterizations of the saturation class as well as a Hoeffding-type result concerning the rate $O(n^{-1/2})$.

Introduction and Results. This note is concerned with the global approximation of integrable functions $f\in L^p:=L^p[0,1]$, $1\leq p<\infty$, by the Kantorovitch polynomials

$$K_n(f;x):=(n+1)\sum_{k=0}^{n}\int_{I_k} f(t)\,dt\,p_{k,n}(x),\quad p_{k,n}(x):=\binom{n}{k}x^k(1-x)^{n-k},$$

where $x\in[0,1]$, $I_k:=[k/(n+1),\ (k+1)/(n+1)]$, and $n\in N$ ($:=$set of natural numbers). Let $B_n(f;x)=B_nf(x):=\Sigma_{k=0}^{n}f(k/n)\,p_{k,n}(x)$ denote the n-th Bernstein polynomial, and set

$$(1)\qquad \varphi(x):=x(1-x),\quad F(x):=\int_0^x f(t)\,dt,\quad \|g\|_{(BV+L^\infty)}:=\|g\|_{BV}+\|g\|_\infty.$$

Furthermore, define for $0<\delta<1/2$, $J_\delta:=[\delta/(1+\delta),\ 1/(1+\delta)]$

$$(2)\qquad \Delta_\delta^2F(x):=F(x+\delta)-2F(x)+F(x-\delta)\qquad (x\in[\delta,1-\delta]),$$

$$(3)\qquad \Delta_\delta^*F(x):=xF(x-(1-x)\delta)-F(x)+(1-x)F(x+x\delta)\qquad (x\in J_\delta).$$

Then the following (global) saturation result holds true for $p=1$:

Theorem A. *For $f\in L^1$ there are equivalent:*

$$(4)\qquad \|B_{n+1}F-F\|_{BV}=\|K_nf-f\|_1=O(1/(n+1))\qquad (n\to\infty),$$

$$(5)\qquad f\in S_1:=\{f\in L^1;\ f\in AC_{\rm loc}(0,1),\ \varphi f'\in BV\},$$

$$(6)\qquad \|\varphi\Delta_\delta^2F\|_{(BV+L^\infty)[\delta,\,1-\delta]}=O(\delta^2)\qquad (\delta\to 0+),$$

$$(7)\qquad \|\Delta_\delta^*F\|_{(BV+L^\infty)[J_\delta]}=O(\delta^2)\qquad (\delta\to 0+).$$

Here $AC_{\rm loc}(0,1)$ denotes the set of functions that are absolutely continuous on compact subintervals of $(0,1)$, whereas $BV:=BV[0,1]$ is the space of functions of bounded variation on $[0, 1]$ with total variation $\|\cdot\|_{BV}$. Of course, $f\in L^1$ such that $f\in AC_{\mathrm{loc}}(0, 1)$ etc. means that f coincides almost everywhere with a function in $AC_{\mathrm{loc}}(0, 1)$ etc.

The pioneering equivalence $(4)\Leftrightarrow(5)$ is due to Maier [6], while the equivalences to (6), (7) may be found in [2] (see also the literature cited there).

Concerning the equivalence $(4)\Leftrightarrow(5)$, an analogous result for $1<p<\infty$ was proved by Riemenschneider [8] (see also Maier [7]).

Theorem B. *For $f\in L^p, 1<p<\infty$, there are equivalent:*

$$\|K_n f-f\|_p=O(1/(n+1)) \quad (n\to\infty), \tag{8}$$

$$f\in S_p:=\{f\in L^p;\, f, f'\in AC_{\mathrm{loc}}(0, 1) \text{ such that for } h:=\varphi f': \; h(0)=h(1)=0,\; h'\in L^p\}. \tag{9}$$

The first aim of this note is to complete Theorem B along the lines of Theorem A, thus:

Theorem 1. *For $f\in L^p$, $1<p<\infty$, assertion* (8) *is equivalent to*

$$\|(\varphi\Delta_\delta^2 F)'\|_{L^p[\delta, 1-\delta]}=O(\delta^2) \quad (\delta\to 0+), \text{ or} \tag{10}$$

$$\|(\Delta_\delta^* F)'\|_{L^p[J_\delta]}=O(\delta^2) \quad (\delta\to 0+). \tag{11}$$

There still remains the problem of characterizing the classes $D_p(\alpha):=\{f\in L^p;\|K_n f-f\|_p=O((n+1)^{-\alpha})\}$ for the nonsaturated cases $0<\alpha<1$. Here we conjecture that

$$\|(\varphi^\alpha \Delta_\delta^2 F)'\|_{L^p[\delta, 1-\delta]}=O(\delta^{2\alpha}) \quad (\delta\to 0+), \text{ or} \tag{12}$$

$$\|(\varphi^{\alpha-1}\Delta_\delta^* F)'\|_{L^p[J_\delta]}=O(\delta^{2\alpha}) \quad (\delta\to 0+) \tag{13}$$

might provide suitable characterizations for $D_p(\alpha)$ (cf. [2] for the conjecture in L^1). But so far, apart from an abstract characterization of $D_1(\alpha)$ via an appropriate K-functional (cf. [4]), this question remains open.

In the particular case $\alpha=1/2$, however, we do have further information (cf. [3] and the literature cited there).

Theorem C. *One has $V_1\subset D_1(1/2)$, where $V_1:=\{f\in L^1;\, f\in BV_{\mathrm{loc}}(0, 1)$ $\varphi^{1/2}f\in BV\}$; in particular, for $f\in V_1$*

$$\|K_n f-f\|_1\le M(n+1)^{-1/2}\|\varphi^{1/2}f\|_{BV+L^\infty}.$$

Moreover, this order of approximation for elements of V_1 cannot be improved in general.

This in fact improves an earlier result of Hoeffding [5], namely $H:=\{f\in L^1;\, f\in BV_{\mathrm{loc}}(0, 1), \int_0^1\varphi^{1/2}(u)\,|df(u)|<\infty\}\subset D_1(1/2)$, since H is a proper subclass of V_1 (cf. [3]).

The second aim of this note is to establish an analogue of Theorem C for the case $p>1$.

Theorem 2. *Let $1<p\le\infty$. Then $V_p\subset D_p(1/2)$, where $V_p:=\{f\in L^p;$ $f\in AC_{\mathrm{loc}}(0, 1)$ such that for $h:=\varphi^{1/2}f$: $h\in AC[0, 1]$, $h(0)=h(1)=0$, $h'\in L^p\}$; in particular, for $f\in V_p$*

$$\|K_n f-f\|_p\le M(n+1)^{-1/2}\|(\varphi^{1/2}f)'\|_p. \tag{14}$$

The following sections provide the proofs of Theorems 1, 2 together with some auxiliary results (cf. La. 1, 2) and more detailed discussions.

Proof of Theorem 1. For $f \in AC_{loc}(0,1)$ there holds the representation (cf. (1), (3))

$$\Delta_\delta^* F(x) = \varphi(x) \int_0^\delta \int_0^{\delta-t} f'(x(1+s+t)-s)\, ds\, dt \quad (x \in J_\delta), \tag{15}$$

and therefore for continuously differentiable f

$$\lim_{\delta \to 0+} \delta^{-2} \Delta_\delta^* F(x) = \varphi(x) f'(x)/2 \quad (x \in (0,1)), \tag{16}$$

uniformly on compact subintervals of (0, 1). This asymptotic expansion shows the strong relation to the Voronovskaya operator of the Kantorovitch polynomials, given by $(\varphi f')'/2$.

Proof of (9)$\Leftrightarrow$(11). First let $f \in S_p$, $1 < p \leq \infty$, and therefore $f' \in L^p$ (cf. [8]). We show that

$$\|(\Delta_\delta^* F)'\|_{L^p[J_\delta]} \leq M\delta^2 [\|f'\|_p + \|(\varphi f')'\|_p]. \tag{17}$$

In view of (15) one has

$$\|(\Delta_\delta^* F)'\|_{L^p[J_\delta]} \leq \int_0^\delta \int_0^{\delta-t} \left\|\frac{d}{dx}[\varphi(x) f'(x(1+s+t)-s)]\right\|_{L^p[J_\delta]} ds\, dt. \tag{18}$$

With the notation $y = y(x, s, t) := x(1+s+t) - s$ there follows

$$\frac{d}{dx}[\varphi(x) f'(y)] = \varphi'(x) f'(y) + (\varphi(x)/\varphi(y))\, y' [(\varphi f')'(y) - \varphi'(y) f'(y)].$$

Since $\|\varphi'\|_\infty = 1$, $y' = 1+s+t \leq 1+h < 2$, this implies that

$$\int_0^\delta \int_0^{\delta-t} \left\|\frac{d}{dx}[\varphi(x) f'(x(1+s+t)-s)]\right\|_{L^p[J_\delta]} ds\, dt$$

$$\leq \|f'\|_p\, \delta^2/2 + 2[\|f'\|_p + \|(\varphi f')'\|_p] \int_0^\delta \int_0^{\delta-t} \|\Phi_{s,t}\|_{L^\infty[J_\delta]}\, ds\, dt,$$

where $\Phi_{s,t}(x) := \varphi(x)/\varphi(x(1+s+t)-s)$. For (s, t) such that $0 < t \leq \delta$, $0 < s \leq \delta - t$ one has $J_\delta \subset (x_1, x_2)$, where $x_1 = s/(1+s+t)$, $x_2 = (1+s)/(1+s+t)$ are the poles of $\Phi_{s,t}$. In view of

$$\frac{d}{dx}\Phi_{s,t}(x) = \frac{t(1+t)x^2 - s(1+s)(1-x)^2}{\varphi^2(x(1+s+t)-s)},$$

the function $\Phi_{s,t}$ has exactly one minimal point in (x_1, x_2), and thus attains its maximal value on J_δ at one of the endpoints, where

$$\Phi_{s,t}\left(\frac{\delta}{1+\delta}\right) = \frac{\delta}{(\delta+\delta t - s)(1+s-\delta t)} \leq \frac{\delta}{(\delta+\delta s - t)(1+t-\delta s)} = \Phi_{s,t}\left(\frac{1}{1+\delta}\right)$$

iff $s \leq t$. Hence we conclude that

$$\int_0^\delta \int_0^{\delta-t} \|\Phi_{s,t}\|_{L^\infty[J_\delta]}\, ds\, dt = \int_0^{\delta/2} \int_t^{\delta-t} \|\Phi_{s,t}\|\, ds\, dt + \int_0^{\delta/2} \int_s^{\delta-s} \|\Phi_{s,t}\|\, dt\, ds \tag{19}$$

$$= 2\delta \int_0^{\delta/2} \int_t^{\delta-t} \frac{ds\, dt}{(\delta+\delta t - s)(1+s-\delta t)} \leq 2\delta \int_0^{\delta/2} \int_t^{\delta-t} \frac{ds\, dt}{\delta + \delta t - s} = \frac{2\delta^2}{1-\delta} \log\left(\frac{2}{1+\delta}\right) \leq (2 \log 2)\, \delta^2.$$

This proves (17) with $M = 4 \log 2 + 1/2$.

To prove the inverse part, define for $n \in N$

$$F_n(x) := \begin{cases} 2n^2 (\Delta^*_{1/n} F)'(x), & x \in J_{1/n}, \\ 0, & x \notin J_{1/n}. \end{cases}$$

By (11) the sequence $\{F_n\}_{n=1}^{\infty}$ is uniformly bounded in L^p. Following standard lines (cf. [2; 6; 7] and the literature cited there), consider the bilinear functional $L_n(f, \psi) := \int_0^1 \psi(x) F_n(x)\, dx$ for $f \in L^p$, $\psi \in C_0^3[0, 1]$ (:= set of three times continuously differentiable functions with compact support in $(0, 1)$). Then weak* compactness yields the existence of a function $h \in L^p$ and a subsequence $\{n_k\} \subset N$ such that

$$\lim_{k\to\infty} L_{n_k}(f, \psi) = \int_0^1 \psi(x) h(x)\, dx. \tag{20}$$

For sufficiently large n (such that $\mathrm{supp}\,(\psi) \subset J_{1/n}$) one has

$$L_n(f, \psi) = -\int_0^1 2n^2 \Delta^*_{1/n} F(x) \psi'(x)\, dx. \tag{21}$$

In view of (16) this implies for $g \in C^1[0, 1]$

$$\lim_{n\to\infty} L_n(g, \psi) = -\int_0^1 \varphi(x) g'(x) \psi'(x)\, dx = \int_0^1 g(x)(\varphi\psi')'(x)\, dx. \tag{22}$$

Next we show that the sequence $\{L_n(\cdot, \psi)\}$ is uniformly bounded on L^p. By (21) one has for $\mathrm{supp}\,(\psi) \subset (2/(n+1), 1-2/(n+1))$ that

$$|L_n(f, \psi)| = \Big| \int_{1/(n+1)}^{n/(n+1)} 2n^2 \Delta^*_{1/n} F(x) \psi'(x)\, dx \Big|$$

$$\le 2n^2 \Big(\frac{n}{n+1}\Big)^2 \int_{1/n}^{1-1/n} |F(x)| \, \Big| \Big(x+\frac{1}{n}\Big) \psi'\Big(\frac{nx+1}{n+1}\Big) - \Big(\frac{n+1}{n}\Big)^2 \psi'(x) + \Big(\frac{n+1}{n} - x\Big) \psi'\Big(\frac{nx}{n+1}\Big) \Big| \, dx$$

$$\le M \|f\|_p \{\|\psi'\|_\infty + \|\psi''\|_\infty + \|\psi'''\|_\infty\}.$$

Therefore, by the Banach-Steinhaus theorem, (22) holds for all $f \in L^p$, so that (20), (22) deliver the integral equation $\int_0^1 f(x)(\varphi\psi')'(x)\, dx = \int_0^1 \psi(x) h(x) dx$. This implies that $f \in S_p$ (cf. [6; 7]). □

Proof of (9)⇔(10). This proof is quite analogous to the one above. In fact, for (9) ⇒ (10) one uses the standard representation

$$\Delta^2_\delta F(x) = \int_{-\delta/2}^{\delta/2} \int f'(x+s+t)\, ds\, dt$$

and a suitable estimation of $\Phi_{s,t}(x) := \varphi(x)/\varphi(x+s+t)$, whereas for (10)⇒(9) one considers the corresponding bilinear functional $L_n(f, \psi)$ based upon

$$F_n(x) := \begin{cases} n^2 (\varphi \Delta^2_{1/n} F)'(x), & x \in [1/n, 1-1/n], \\ 0, & x \notin [1/n, 1-1/n]. \end{cases}$$

See also the treatment given in [2] for $p=1$. □

Let us note that, corresponding to (9) ⇒ (11), one may use the representation (15) to establish (5) ⇒ (7), which in fact would simplify the argument given in [2].

Proof of Theorem 2. Before we turn to the actual proof of Theorem 2, let us add some remarks concerning $D_p(\alpha)$ for $0<\alpha<1$. The following lemma may be considered as an extension of Theorem 1 in [3] from $p=1$ to $p>1$ (for related material in case $\alpha=1$ see [6;8]).

Lemma 1. *Let* $0<\alpha<1$, $1\leq p\leq\infty$, $1/p+1/q=1$. *Then for the particular function* $f_\alpha(x):=x^{\alpha-1}$ *one has*

(23) $$\| x^{1/q}[K_n(f_\alpha;x)-f_\alpha(x)]\|_p=O((n+1)^{-\alpha}).$$

Proof. The following identity was shown in [3]

$$K_n(f_\alpha;x)-f_\alpha(x)=(n+1)^{1-\alpha}\sum_{k=0}^{n} a_{k,n}(\alpha)p_{k,n}(x)-\sum_{k=n+1}^{\infty}\binom{\alpha-1}{k}(x-1)^k,$$

where $a_{k,n}(\alpha)$ are certain coefficients with

$$0\leq a_{k,n}(\alpha)\leq\frac{1}{\alpha}[(k+1)^\alpha-k^\alpha]-(k+1)^{\alpha-1}\qquad(0\leq k\leq n),$$

$$=\frac{1}{\alpha}(k+1)^{\alpha-2}\sum_{j=0}^{\infty}\binom{\alpha}{j+2}\frac{(-1)^{j+1}}{(k+1)^j}\qquad(1\leq k\leq n).$$

Hence one has

(24) $$a_{k,n}(\alpha)\leq M_\alpha(k+1)^{\alpha-2}\qquad(0\leq k\leq n).$$

Let us also note that (for (27) see [3])

(25) $$xp_{k,n}(x)=\frac{k+1}{n+1}p_{k+1,\,n+1}(x),$$

(26) $$\max_{x\in[0,1]} x(1-x)^k=\frac{1}{k+1}\Big(1-\frac{1}{k+1}\Big)^k\leq\frac{1}{k+1},$$

(27) $$\sum_{k=n+1}^{\infty}\frac{1}{k+1}\left|\binom{\alpha-1}{k}\right|=(n+1)^{-\alpha}\sum_{k=0}^{n}a_{k,n}(\alpha)\leq\frac{1}{\alpha}(n+1)^{-\alpha}.$$

Using (24)—(27), the case $p=\infty$ in (23) follows since

$$|x[K_n(f_\alpha;x)-f_\alpha(x)]|\leq(n+1)^{1-\alpha}\sum_{k=0}^{n}a_{k,n}(\alpha)xp_{k,n}(x)+\sum_{k=n+1}^{\infty}\left|\binom{\alpha-1}{k}\right|x(1-x)^k$$

$$\leq M_\alpha(n+1)^{-\alpha}\sum_{k=0}^{n}(k+1)^{\alpha-1}p_{k+1,\,n+1}(x)+\sum_{k=n+1}^{\infty}\frac{1}{k+1}\left|\binom{\alpha-1}{k}\right|$$

$$\leq M_\alpha(n+1)^{-\alpha}\sum_{k=0}^{n}p_{k+1,\,n+1}(x)+\frac{1}{\alpha}(n+1)^{-\alpha}=O((n+1)^{-\alpha}).$$

Note incidentally that the sum corresponding to the first term in case $\alpha=1$ may be estimated analogously using (25); this would simplify the argument in [8]. Since the case $p=1$ was given in [3], Hölder's inequality immediately yields (23) for all $1\leq p\leq\infty$. □

By a change of variable one also has that for $g_\alpha(x):=(1-x)^{\alpha-1}$

(28) $$\|(1-x)^{1/q}[K_n(g_\alpha;x)-g_\alpha(x)]\|_p=O((n+1)^{-\alpha}).$$

Moreover, it is possible to extend (23), (28) to more general functions, similarly to Theorem 1 of [3].

Lemma 2. *Let* $0<\alpha<1$, $1\leq p\leq\infty$, $1/p+1/q=1$, *and suppose that* $\lambda, \rho\in AC[0, 1]$ *are monotonically increasing functions such that* $\lambda(1)=\rho(0)=0$ *and* $\lambda', \rho'\in L^p$. *Then for* $f_\lambda, f_\rho\in L^1\cap AC_{\text{loc}}(0, 1)$ *such that* $x^{1/q}f_\lambda(x)$, $(1-x)^{1/q}f_\rho(x)\in L^p$ *and* $\varphi^{2-\alpha}f'_\lambda=\lambda$, $\varphi^{2-\alpha}f'_\rho=\rho$, *respectively, there holds*

$$\| x^{1/q}[K_n(f_\lambda; x)-f_\lambda(x)]\|_p=O((n+1)^{-\alpha}), \tag{29}$$

$$\|(1-x)^{1/q}[K_n(f_\rho; x)-f_\rho(x)]\|_p=O((n+1)^{-\alpha}). \tag{30}$$

Proof. We shall only discuss (29). In view of $\varphi^{2-\alpha}f'_\lambda=\lambda$ and the monotonicity of λ one has (compare [3] for $p=1$ and [6] for $\alpha=1$)

$$f_\lambda(x)-f_\lambda(t)=\int_t^x\frac{\lambda(u)du}{\varphi^{2-\alpha}(u)}\leq M_\alpha|\lambda(x)|[f_\alpha(x)-f_\alpha(t)+g_\alpha(t)-g_\alpha(x)].$$

Therefore the proof follows from (23), (28) since $|\lambda(x)|\leq\int_x^1|\lambda'(u)|du\leq(1-x)^{1/q}\|\lambda'\|_p$. □

Let us remark that (23), (28) imply $f_\alpha, g_\alpha\in D_p(\alpha)$ only for $p=1$, where these estimates were shown to be best possible (cf. [3]). Here we do not attempt to show the sharpness of (23), (28) for all $1<p\leq\infty$, though we believe that it is true. Another consequence of this behaviour is the fact that for $p>1$ the conditions $h(0)=h(1)=0$ were added in the definition of S_p and V_p, though it was shown in [2; 3] that they may be replaced by $h\in L^\infty$ in case $p=1$.

Proof of Theorem 2. Here we follow the proof for $p=1$ given in [3], which in turn was a suitable modification of the method of proof for (9) $\Rightarrow$ (8) (cf. [8]). Setting $h:=\varphi^{1/2}f$ and $g(x):=\arcsin(2x-1)$, one has $h'\in L^p$ for $f\in V_p$ and $g'=\varphi^{-1/2}$. From [3] we recall the expansion

$$B_n(F; x)-F(x)=h(x)[B_n(g; x)-g(x)]+B_n\Big(\int_t^x[g(u)-g(t)]h'(u)\,du; x\Big). \tag{31}$$

Therefore we obtain for $f\in V_p$, $1<p\leq\infty$,

$$\|K_{n-1}f-f\|_p=\Big\|\frac{d}{dx}\{B_n(F; x)-F(x)\}\Big\|_p\leq\|h(x)[K_{n-1}(\varphi^{-1/2}; x)-\varphi^{-1/2}(x)]\|_p \tag{32}$$

$$+\Big\|\sum_{k=0}^{n}\int_{k/n}^{x}[g(u)-g(\tfrac{k}{n})]h'(u)\,du\,p'_{k,n}(x)\Big\|_p=:\|I_1\|_p+\|I_2\|_p.$$

In view of $h(0)=h(1)=0$ there follows by Hölder's inequality that $|h(x)|\leq x^{1/q}\|h'\|_{p'}$, $|h(x)|\leq(1-x)^{1/q}\|h'\|_p$. Using $\varphi^{-1/2}(x)=\sqrt{x/(1-x)}+\sqrt{(1-x)/x}$ one has by (29), (30) with $\lambda(x)=(x-1)/2$, $\rho(x)=x/2$ that

$$\|I_1\|_p\leq\|h'\|_p\|x^{1/q}[K_{n-1}(\sqrt{(1-t)/t}; x)-\sqrt{(1-x)/x}]\|_p \tag{33}$$

$$+\|h'\|_p\|(1-x)^{1/q}[K_{n-1}(\sqrt{t/(1-t)}; x)-\sqrt{x/(1-x)}]\|_p\leq M\|h'\|_p n^{-1/2}.$$

Concerning I_2, let us first consider the case $p=\infty$. One has

$$\|I_2\|_\infty\leq\|h'\|_\infty\Big\|\frac{n}{\varphi(x)}\sum_{k=0}^{n}\Big|\frac{k}{n}-x\Big|\int_{k/n}^{x}\int_{k/n}^{u}\varphi^{-1/2}(s)ds\,du\,p_{k,n}(x)\Big\|_\infty. \tag{34}$$

Since for $0\leq\alpha\leq1$, $x\in[0, 1]$, $t\in(0, 1)$ (cf. [1]) $\int_t^x\int_t^u\varphi^{-\alpha}(s)ds\,du\leq\varphi^{-\alpha}(t)(t-x)^2$, there follows, using elementary estimates for the cases $k=0$, n,

$$\| I_2 \|_\infty \leqq \| h' \|_\infty [Mn^{-1/2} + \| A \|_\infty], \quad A(x) := \frac{n}{\varphi(x)} \sum_{k=1}^{n-1} \left| \frac{k}{n} - x \right|^3 \varphi^{-1/2}\left(\frac{k}{n}\right) p_{k,n}(x).$$

With $B_n^0(f;x) := \sum_{k=1}^{n-1} f(k/n) p_{k,n}(x)$ one has for $x \in [1/n, 1-1/n]$ that $n\varphi(x) \geqq 1/2$ and therefore (cf. [1])

$$A(x) \leqq n\varphi^{-1}(x) [B_n((t-x)^4; x)]^{3/4} [B_n^0(\varphi^{-2}(t); x)]^{1/4}$$
$$\leqq n\varphi^{-1}(x) [M\varphi^2(x)/n^2]^{3/4} [M\varphi^{-2}(x)]^{1/4} = Mn^{-1/2}.$$

For $x \in [0, 1/n]$ (and correspondingly for $x \in [1-1/n, 1]$), $x^{k-1}(1-x)^{n-k-1}$ is monotonically increasing and hence

$$A(x) \leqq Mn^{-1/2} + n \sum_{k=2}^{n-1} \left(\frac{k}{n}\right)^3 \varphi^{-1/2}\left(\frac{k}{n}\right) \binom{n}{k} \left(\frac{1}{n}\right)^{k-1} (1-1/n)^{n-k-1}$$
$$\leqq Mn^{-1/2} + MnB_{n-1}^0 (t^{3/2}(1-t)^{-1/2}; 1/n)$$
$$\leqq Mn^{-1/2} + Mn[B_{n-1}(t^4; \frac{1}{n})]^{1/2} [B_{n-1}^0(\varphi^{-1}(t); 1/n)]^{1/2}$$
$$\leqq Mn^{-1/2} + Mn [Mn^{-4}]^{1/2} [M\varphi^{-1}(1/n)]^{1/2} = Mn^{-1/2}.$$

This shows $\| I_2 \|_\infty \leqq M \| h' \|_\infty (n+1)^{-1/2}$.

To extend the argument to all values $1 \leqq p \leqq \infty$, one employs the theorem of Riesz-Thorin. In fact, the linear operator

$$T_n f(x) := (n+1)^{1/2} \frac{d}{dx} B_n \left(\int_t^x [g(u) - g(t)] f(u)\, du; x \right)$$

is uniformly bounded on L^1, since for $h \in BV$ (cf. [3]) $\| B_n(\int_t^x [g(u)-g(t)]\, dh(u); x) \|_{BV} \leqq M(n+1)^{-1/2} \| h \|_{BV}$, as well as on L^∞, which follows by (34) and (cf. [3])

$$T_n f(x) = (n+1)^{1/2} \{ I_2 + f(x) [g(x) - B_n(g; x)] \}, \| B_n(g(x) - g(t); x) \|_\infty \leqq M(n+1)^{-1/2}.$$

Therefore $T_n f$ is bounded on L^p for all $1 \leqq p \leqq \infty$, and the proof then follows by taking $f = h'$. □

Added in Proof: Let us note that the problem of characterizing the classes $D_p(\alpha)$, $0<\alpha<1$, via certain second differences has meanwhile been solved by V. Totik (Széged) in several papers which will appear in *Acta Math. Acad. Sci. Hung.*, *Analysis*, *Analysis Math.*, *J. Approx. Theory*, *Pacific J. Math.*, etc.

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