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ON THE POLYNOMIAL APPROXIMATION OF sign $(x-a)$

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Summary. Let a, $-1<a<1$, be a fixed number and $k>0$ be such that $[a-k, a+k]\subset[-1, 1]$ and let $n\geqq 0$ be a fixed integer. We are interested in finding the smallest $d_n=d_n(n, a, k)$ such that there exists a polynomial $p_n(z)$, $\deg p_n\leqq n$ whose graph is contained in the set $D_n=\Delta_{n1}\cup\Delta_{n2}\cup\Delta_{n3}$, where $\Delta_{n1}=\{(x, y): -1\leqq x\leqq a-k,\ -1-d_n\leqq y\leqq -1+d_n\}$, $\Delta_{n2}=\{(x, y): a-k\leqq x\leqq a+k,\ -1-d_n\leqq y\leqq 1+d_n\}$, $\Delta_{n3}=\{(x, y): a+k\leqq x\leqq 1,\ 1-d_n\leqq y\leqq 1+d_n\}$. The following statement holds.

Theorem. *Let $w(z)$ be a linear-fractional transformation mapping the segments $[-1, a-k]$ and $[a+k, 1]$ onto the $[-1, \alpha]$ and $[\alpha, 1]$, $\alpha=\alpha(a, k)$. If $d=w(\infty)$ and $\varphi(w)=2w^2/(1-\alpha^2)+(1-\alpha^2)^{-1}2((w^2-1)(w^2-\alpha^2))^{1/2}+(\alpha^2-1)^{-1}(\alpha^2+1)$ then*

$$\overline{\lim}\, d_n^{1/n}\leqq(\varphi(d)-\varphi(0))/(1-\varphi(d)\varphi(0))\leqq e^{Ak},$$

where $A=-2/(1-a^2)+2a^2/(1-a^2)(1+(1-a^2)^{1/2})$.

1. Let a, $-1<a<1$, be a fixed number, $k>0$ be such that $[a-k, a+k]\subset[-1, 1]$ and $n\geqq 0$ be a fixed integer. We are interested in finding the smallest $d_n=d(n, a, k)$ such that there exists a polynomial $p_n(z)$, $\deg p_n\leqq n$ whose graph is contained in the set $D_n=\Delta_{n1}\cup\Delta_{n2}\cup\Delta_{n3}$, where

$$\Delta_{n1}=\{(x, y):\ -1\leqq x\leqq a-k,\quad -1-d_n\leqq y\leqq -1+d_n\},$$

$$\Delta_{n2}=\{(x, y):\ a-k\leqq x\leqq a+k,\quad -1-d_n\leqq y\leqq 1+d_n\},$$

$$\Delta_{n3}=\{(x, y):\ a+k\leqq x\leqq 1,\quad 1-d_n\leqq y\leqq 1+d_n\}.$$

It is clear that this problem is a weak variant of the problem of approximation of the function sign $(x-a)$ by polynomials of degree $\leqq n$ with respect to the Hausdorff distance (for the definition and the main properties of the Hausdorff distance we refer to [1]).

In order to solve the above problem, at least assymptotically, it is possible to use successfully the method of approximation of analytic functions by polynomials worked out by J. Walsh [2]. Namely, let $E\subset \boldsymbol{C}$ (we denote by $\boldsymbol{C}$ the complex plane and by $\bar{\boldsymbol{C}}=\boldsymbol{C}\cup\{\infty\}$ the extended complex plane) be a compact set with a connected complement and let $g(z, \infty)$ be the Green's function of $\boldsymbol{C}\setminus E$. It is well-known [2] that there exists a table of points $\{z_{ns}\}$, $n=1, 2, \ldots, s=1, 2, \ldots, n$, which belong to E and such that

(1) $$\lim \prod_{s=1}^{n} |z-z_{ns}|^{1/n} = \operatorname{cap} E \,.\, \exp(g(z, \infty))$$

holds uniformly on the compact sets of $C\setminus E$ (by $\operatorname{cap} E$ we denote the Chebyshev's constant (the transfinite diameter) of E).

Let f be a function, analytic in a neighbourhood D_f of E and let $\sigma>0$ be such that the level line $\Gamma_\sigma=\{z\in C: g(z, \infty)=\sigma\}$ is contained in D_f. Denote by $\omega_n(z)$ the polynomial

$$\omega_n(z)=\prod_{s=1}^{n}(z-z_{ns})$$

and let $p_n(z)$ be the polynomial of a degree $\leq n-1$ which interpolates $f(z)$ at the points $z_{n1}, z_{n2}, \ldots, z_{nn}$. Then from (1) and the Hermit's interpolating formula

(2) $$f(z)-p_n(z)=\frac{1}{2\pi i}\int_{\Gamma_\sigma}\frac{f(t)\cdot\omega_n(z)}{\omega_n(t)(t-z)}\,dt$$

we could easily find

(3) $$\overline{\lim}\, \|f-p_n\|_E^{1/n}\leq e^{-\sigma},$$

where $\|\cdot\|_E$ is the sup-norm on E. The theorem of Bernstein-Walsh yields that if σ is the largest number such that $\Gamma_\sigma\subset D_f$ then $e^{-n\sigma}$ gives the sharp order of the uniform approximation of the f by polynomials on E.

2. Let $f(z)$ be a holomorphic in $C\setminus E$ function with the following properties: a) $|f(z)|>1$, b) $|f(z)|\to 1$, $z\to\zeta\in E$, $z\in C\setminus E$ and c) $z=\infty$ is a pole of the f of multiplicity s. If

$$f(z)=a_s z^s+a_{s-1}z^{s-1}+\ldots+\frac{b_1}{z}+\frac{b_2}{z^2}+\ldots$$

is the Laurant's series of f around $z=\infty$, then the function $g(z) = (1/s)\ln|f(z)|$ has the following properties:

1) $g(z)$ is harmonic in $C\setminus E$.
2) $g(z)\to 1$, $z\to E$, $z\in C\setminus E$.
3) $g(z)-\ln|z|$ is harmonic in $C\setminus E$.

Since these properties determine the Green's function uniquely we can claim that $g(z)=g(z, \infty)$. Thus in order to find the Green's function of $C\setminus E$ it is sufficient to find a function f with the properties mentioned above.

Let $w(z)$ be a linear-fractional transformation mapping the segments $[-1, a-k]$ and $[a+k, 1]$ onto the segments $[-1, \alpha]$ and $[\alpha, 1]$, $\alpha=\alpha(a, k)>0$. It is easy to calculate that there exist two such transformations $w(z)=\pm(z-\beta)/(1-\beta z)$, where $\beta=\beta(a, k)=(a^2-k^2+1-((a^2-k^2+1)^2-4a^2)^{1/2})2a$. In order to fix the situation we shall choose the sign in such a way that $d=d(a, k)=w(\infty)=\pm 1/\beta>1$. Besides it is easy to calculate that $\alpha=\alpha(a, k)=2k/(1+k^2-a^2+((1+k^2-a^2)^2-4k^2)^{1/2})$. It is clear that $\alpha(a, k)$ and $\beta(a,k)$ are differentiable functions of k near $k=0$ and that $\beta(a, 0)=a$, $\alpha(a, 0)=0$.

Lemma 1. *Let* $E=[-1,\ a-k]\cup[a+k,1]$, $a\neq0$ *and let* $w=w(z)$ $\beta(a,k)$ *and* $\alpha(a,k)$ *be the functions defined above. Let us define the functions*

$$\varphi(w)=\frac{2w^2}{1-\alpha^2}+\frac{2}{1-\alpha^2}((w^2-1)(w^2-\alpha^2))^{1/2}+\frac{\alpha^2+1}{\alpha^2-1}, \tag{4}$$

$$\xi(\varphi)=(1-c\varphi)/(\varphi-c),\quad c=c(a,k)=\varphi\circ w(\infty). \tag{5}$$

Then the Green's function of the $C\setminus E$ *coincides with* $\ln|f(z)|$, *where* $f(z)=\xi\circ\varphi\circ w(z)$. *We choose this branch of the square root in the definition of* $\varphi(w)$, *which is* >0 *for* $w=u+i0$, $|u|>1$.

It is easy to construct $f(z)$ as a mapping of $C\setminus E$ onto $|\xi|>1$ in such a way that $\infty\to\infty$. The only step in this construction, where the conformality fails, is $w\to w^2$. We'll show that around $z=\infty$

$$f(z)=\varepsilon\frac{1-c^2}{\varphi'(d)(1-d^2)}\cdot z+f_0+\frac{f_1}{z}+\dots, \tag{6}$$

where $d=w(\infty)$, $\varepsilon=+1$, if $a<0$ and $\varepsilon=-1$, if $a>0$. In fact

$$\lim_{z\to\infty}(f(z)/z)=(1-c^2)/\lim_{z\to\infty}z(\varphi(w(z))-\varphi(w(\infty))).$$

As $\varphi(w)-\varphi(d)=\varphi'(d)(w-d)+\varphi''(w-d)^2/2+\dots$ and $w(z)-d=\varepsilon(1-\beta^2)/\beta(1-z\beta)$, then $\lim z(\varphi(w(z))-\varphi(w(\infty)))=\varphi'(d)\lim z(w(z)-w(\infty))=\varepsilon\varphi'(d)\times(1-d^2)$. We see from (6) that $z=\infty$ is a pole of $f(z)$ of multiplicity 1 and hence $g(z,\infty)=\ln|f(z)|$.

Corollary 1. *If* $E=[-1,a-k]\cup[a+k,1]$ *then* $\operatorname{cap}E=\varphi'(d)(1-d^2)/(1-c^2)$, $c=\varphi(d)$.

Remark. The case $a=0$ can be considered in the same way (in fact much easier). In this case

$$f(z)=\frac{2z^2}{1-k^2}+\frac{2}{1-k^2}((z^2-1)(z^2-k^2))^{1/2}+\frac{k^2+1}{k^2-1},$$

$$g(z,\infty)=\frac{1}{2}\ln|f(z)|,\quad \operatorname{cap}E=\sqrt{1-k^2}/2.$$

Lemma 2. *The maximum of the function* $|f(x)|=\exp g(x,\infty)$ *in* $[a-k,\ a+k]$ *is equal to* $(1-\varphi(0)c)/(c-\varphi(0))$, *where* $c=c(a,k)=\varphi(d)$, $d=w(\infty)$.

Proof. It is easy to verify that $|f(x)|=-f(x)$ in the segment $[a-k,a+k]$. Since $f'(z)=0$ if and only if $z=\beta$, then $\max f(x)=-f(\beta)=(1-c\varphi(0))(c-\varphi(0))$, $x\in[a-k,a+k]$.

Theorem 1. *Let* $h(z)$ *be analytic in the open set* $C\setminus\{z:\operatorname{Re}z=\beta\}$. *Then there exists a sequence* $p_n(z)$ *of polynomials,* $\deg p_n\leq n-1$ *such that*

$$\overline{\lim}\,\|h-p_n\|_E^{1/n}\leq(c-\varphi(0))/(1-c\varphi(0)). \tag{7}$$

Proof. Let $\{z_{ns}\}$ $n=1,2,\dots,s=1,2,\dots,n$ be a table of points which belong to E such that

$$\lim\prod_{s=1}^{n}|z-z_{ns}|^{1/n}=\operatorname{cap}Ee^{g(z,\infty)}=\operatorname{cap}E\,|f(z)| \tag{8}$$

holds uniformly on the compact sets of $C \setminus E$. It follows from Lemma 2 that the level line $\Gamma = \{z : |f(z)| = (1 - c\varphi(0))/(c - \varphi(0))\}$ lies in $C \setminus \{z : \operatorname{Re} z = \beta\}$ with the only exception of the point $z = \beta$. Then the estimation (7) follows easily from (2) and (8).

We shall find the solution of the problem we are dealing with as a consequence of Theorem 1. Let us note that our estimation is only an asymptotic one and we don't know the exact value of d_n for every fixed n.

Corollary 2. *If d_n is the sequence introduced at the beginning of the paper then*

(9) $$\limsup d_n^{1/n} \leq (c - \varphi(0))/(1 - c\varphi(0)), \quad c = \varphi(w(\infty)).$$

Proof. Define the function $h(z)$ equal to -1 if $\operatorname{Re} z < \beta$ and to 1 if $\operatorname{Re} z > \beta$. Then $h(z)$ is analytic in $\{z \in C : \operatorname{Re} z \neq \beta\}$. Applying (7), we find that the graph of the polynomials $p_n(x)$, $n \geq n_0$ for $x \in [-1, a-k] \cup [a+k, 1]$ lies in the set $D_1^n \cup D_2^n$, where $D_1^n = \{(x, y) : -1 \leq x \leq a-k, \quad -1 - q^n \leq y \leq -1 + q^n\}$, and $D_2^n = \{(x, y) : a+k \leq x \leq 1, 1 - q^n \leq y \leq 1 + q^n\}$, and q is an arbitrary number larger than $(c - \varphi(0))/(1 - \varphi(0)c)$. The only thing which remains to be verified is that $|p_n(x)| \leq 1 + q^n$ in the $[a-k, a+k]$. We shall show that even

(10) $$|p_n(x)| \leq 1 \quad \text{in} \quad a - k \leq x \leq a + k,$$

which is an easy consequence from the Rolle's theorem. Really, if we denote by s_n the number of the points z_{ns}, which lie in $[-1, a-k]$, then $p_n(z_{ns})$ coincides with -1 for $s \leq s_n$ and with 1 for $s > s_n$. Thus p_n' has at least one zero in each of the intervals $(z_{ns}, z_{n,s-1})$ $s \leq s_n$ and $s \geq s_n + 2$, and hence $p_n' \neq 0$ in $[z_{n,s_n}, z_{n,s_n+1}]$. Consequently $p_n(x)$ is monotone in (z_{n,s_n}, z_{n,s_n+1}) and (10) follows.

Lemma 3. *Let $a \neq 0$ and $\varepsilon > 0$ be arbitrary numbers. Then there exists a $k_0 = k(a, \varepsilon)$ such that for $k \leq k_0$*

(11) $$\theta(a, k) = \frac{c(a, k) - \varphi(0)}{1 - \varphi(0)c(a, k)} \leq e^{(A+\varepsilon)k},$$

where

$$A = \frac{-2}{1 - a^2} + \frac{2a^2}{(1 - a^2)(1 + \sqrt{1 - a^2})}.$$

Proof. Denote by $e(a . k) = ((1+k)^2 - a^2)/((1-k)^2 - a^2)^{1/2}$. Then $\varphi(0) = (\alpha + 1)/(\alpha - 1) = -e(a, k)$ and hence $\theta(a, k) = (c + e)/(1 + ce)$. Using the equalities $e(a, 0) = 1$, $e_k'(a, 0) = 2/(1 - a^2)$, $c(a, 0) = -1 + 2(1 + (1 - a^2)^{1/2})/a^2$ it is easy to show that $\theta'(a, 0)/\theta(a, 0) = A$ and hence in some neighbourhood of $k = 0$ the function $\ln \theta(a, k) - k(A + \varepsilon)$ will be negative and thus (11) follows.

Corollary 3. *If $a \neq 0$ and $\varepsilon > 0$, then for $k \leq k(a, \varepsilon)$ $\overline{\lim}\, d_n^{1/n} \leq \exp k(-2 + \varepsilon)$. If $a = 0$ then $\overline{\lim}\, d_n^{1/n} \leq e^{-k}$.*

P r o o f. If $a \neq 0$ then the statement follows from Corollary 2 (11) and the inequality $A \leqq -2$. The case $a=0$ can be easily attacked directly. In this case

$$\theta(a, k) = \exp(2^{-1} \ln((1-k)/(1+k)))$$

(cf. Remark 1) and the inequality $e^{-k} \geqq \theta(0, k)$ is clear.

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