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AN EXTREMAL PROBLEM FOR POLYNOMIALS

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Summary. Let $\varphi(x)$ be a strictly convex, increasing function in $[0, \infty]$. We prove that

$$\sup\left\{\int_{-1}^{1}\varphi(|f'(x)|)dx : f\in\pi_n,\ \|f\|_{C[-1,1]}\leqq 1\right\}=\int_{-1}^{1}\varphi(|T_n'(x)|)dx,$$

where $T_n(x)$ is the Chebyshev polynomial. As a corollary we obtain the inequality

$$\|f'\|_p\leqq\|T_n'\|_p\,\|f\|_\infty\quad(1<p<\infty)$$

for each $f\in\pi_n$.

Let π_n denote the set of algebraic polynomials of degree not exceeding n. Set

$$B(\pi_n) := \{f\in\pi_n : \|f\|\leqq 1\},$$

where $\|f\|$ is the uniform norm of f in the interval $[-1, 1]$. We prove here the following extremal property of the famous Chebyshev polynomials $T_n(x)=\cos(n\arccos x)$.

Theorem 1. *Let $\varphi(x)$ be a strictly convex, increasing function in $[0, \infty)$. Then, for each natural number n, the quantity*

$$\sup\left\{\int_{-1}^{1}\varphi(|f'(x)|)dx : f\in B(\pi_n)\right\}\tag{1}$$

is attained if and only if $f=\pm T_n$.

Proof. Without loss of generality we may assume that $\varphi(x)$ satisfies the requirements

$$\varphi(0)=0,\qquad \varphi'(0)=0,\tag{2}$$

since every convex increasing function in $[0, \infty)$ can be presented in the form $\varphi(x)=\varphi(0)+Ax+\tilde{\varphi}(x)$ with $A\geqq 0$, $\tilde{\varphi}$ satisfying (2), and the theorem is trivially proved in the case $\varphi(x)=B+Ax$, $A>0$.

Denote, for simplicity,

$$L(f) := \int_{-1}^{1}\varphi(|f'(x)|)dx.$$

Let us assume that $f \in B(\pi_n)$ and

(3) $$L(f) = \sup\{L(g) : g \in B(\pi_n)\}.$$

Denote by $\{x_k\}_1^{m-1}$ the distinct zeros of $f'(x)$ in $(-1, 1)$. We shall prove first that

(4) $$|f(x_k)| = 1, \quad k = 0, \ldots, m,$$

where $x_0 = -1$, $x_m = 1$. Indeed, set for convenience $\omega(x) = f'(x)$. Consider the polynomial $f(x) + \varepsilon g_i(x)$, where $g_i(x) = \omega(x)(x^2 - 1)/(x - x_i)$. Put $\sigma_i(\varepsilon) = L(f + \varepsilon g_i)$. We have

$$\sigma_i'(0) = \int_{-1}^{1} \varphi'(|\omega(x)|) \operatorname{sign} \omega(x) \cdot g_i'(x) dx$$

$$= \int_{-1}^{1} \varphi'(|\omega(x)|) \operatorname{sign} \omega(x) \{(\frac{x^2-1}{x-x_i})' \omega(x) + \frac{x^2-1}{x-x_i} \omega'(x)\} dx$$

$$= \int_{-1}^{1} \varphi'(|\omega(x)|) |\omega(x)| (\frac{x^2-1}{x-x_i})' dx + \int_{-1}^{1} \frac{x^2-1}{x-x_i} \{\varphi(|\omega(x)|)\}' dx.$$

A careful integration by parts (over the domain $[-1, x_i - \delta] \cup [x_i + \delta, 1]$ and next taking $\delta \to 0$) shows that

$$\int_{-1}^{1} \frac{x^2-1}{x-x_i} \{\varphi(|\omega(x)|)\}' dx = -\int_{-1}^{1} \varphi(|\omega(x)|) (\frac{x^2-1}{x-x_i})' dx.$$

Then

$$\sigma_i'(0) = \int_{-1}^{1} [\varphi'(|\omega(x)|) |\omega(x)| - \omega(|\varphi(x)|)](1 + \frac{1-x_i^2}{(x-x_i)^2}) dx.$$

But clearly $\varphi'(t)t > \varphi(t)$ if φ satisfies (2). Therefore

(5) $$\sigma_i'(0) > 0, \quad i = 0, \ldots, m.$$

Now suppose that (4) is not valid. Thus, there exists an $i \in \{0, \ldots, m\}$ such that $|f(x_i)| < 1$. Then it is easily seen that

(6) $$\| f + \varepsilon g_i \| = 1 + o(\varepsilon),$$

as $\varepsilon \to 0$. On the other hand, in view of (5), there exists a constant $c > 0$ such that $\sigma_i(\varepsilon) > \sigma_i(0) + c\varepsilon$ for each sufficiently small $\varepsilon > 0$, i. e.

(7) $$L(f + \varepsilon g_i) > L(f) + c\varepsilon.$$

Denote by ψ_ε the polynomial $(f + \varepsilon g_i)/\| f + \varepsilon g_i \|$. Evidently, $\psi_\varepsilon \in B(\pi_n)$. Further, (6) and (7) imply $L(\psi_\varepsilon) > L(f)$ for each sufficiently small $\varepsilon > 0$. This contradicts the assumption that f is an extremal element. Therefore (4) is true.

Denote by Ω_n the set of those polynomials f from the set $\{g \in \pi_n : \| g \| = 1\}$ for which there exist $m+1$ points $\{x_i\}_0^m$, $-1 = x_0 < x_1 < \ldots < x_m = 1$ $(m \in \{1, \ldots, n\})$, such that

(8) $$f(x_k)=(-1)^{m-k}, \quad k=0, \ldots, m,$$
and $f(x)$ is a monotone function in $[x_k, x_{k+1}]$, $k=0, \ldots, m-1$. It follows from (4) that if f satisfies (3) (and f is normalized by the condition $f(\infty)=\infty$) then f must belong to the set Ω_n. In order to complete the proof of the theorem we need only show that

(9) $$L(f)<L(T_n)$$

for each $f\in\Omega_n$, $f\neq T_n$.

Suppose that $f\in\Omega_n$ and $\{x_k\}_0^m$, $-1=x_0<x_1<\ldots<x_m=1$, are the extremal points of f in $[-1, 1]$. There is an $i\in\{0, \ldots, m-1\}$ such that $x_i<0\leq x_{i+1}$. Consider the partition of $[-1, 1]$ into subintervals $[x_0, x_1]$, $\ldots$, $[x_i, 0]$, $[0, x_{i+1}]$, $\ldots$, $[x_{m-1}, x_m]$ which we denote by $I_0, \ldots, I_m$, respectively. Let $\{\theta_k\}_0^n$ be the extremal points of $T_n(x)$ in $[-1, 1]$. Define the points t_1 and t_2 by the conditions

$$t_1\in[\theta_i, \theta_{i+1}], \quad T_n(t_1)=f(0),$$

$$t_2\in[\theta_{i+n-m}, \theta_{i+n-m+1}], \quad T_n(t_2)=f(0).$$

Denote the intervals

$$[\theta_0, \theta_1], \ldots, [\theta_i, t_1], [t_2, \theta_{i+n-m+1}], \ldots, [\theta_{n-1}, \theta_n]$$

by $I_0^*, \ldots, I_m^*$, respectively. We shall use the following interesting property of the Chebyshev polynomials.

Property 1. Let $f\in\Omega_n$, $\alpha\in(-1, 1)$ and $k\in\{0, \ldots, m\}$. Let the points ξ and η be defined by the conditions

$$\xi\in I_k^*, \quad T_n(\xi)=\alpha,$$

$$\eta\in I_k, \quad f(\eta)=\alpha.$$

Then

(10) $$|f'(\eta)|\leq|T_n'(\xi)|$$

and the equality is attained if and only if $f=T_n$.

To prove this property we first observe that $|\eta|<|\xi|$ for $f\neq T_n$ (otherwise the polynomial $T_n(x)-f(x)$ would have more than n zeros, thus $f(x)\equiv T_n(x)$). Then (10) follows immediately from the equalities

$$T_n'(\xi)=n\sin(n\arccos\xi)(1-\xi^2)^{-1/2},$$

$$f'(\eta)=-\tau'(\arccos\eta)(1-\eta^2)^{-1/2},$$

where $\tau(t)=f(\cos t)$, applying the known fact that

$$|n\sin na|>|\tau'(b)|$$

for each trigonometric polynomial $\tau(t)$ of order at most n such that $\cos na=\tau(b)$ and $\|\tau\|_{C[-\pi,\pi]}\leq 1$.

Next we follow an idea used by L. Taikov [4] in the proof of the trigonometric analog of Theorem 1.

Evidently there is an $M>0$ such that $\|f'\|\leq M$ if $f\in B(\pi_n)$. With every $\alpha\in[0, M]$ we associate the function

$$\varphi_\alpha(x):=\begin{cases} 0, & 0\leq x<\alpha, \\ x, & \alpha\leq x\leq M. \end{cases}$$

Divide the interval $[0, M]$ into N equal parts by the points $\alpha_k=kM/N$, $k=0, \ldots, N$. Construct the function $\Phi_N(x)=\Sigma_{k=1}^{N-1}\beta_k\varphi_{\alpha_k}(x)$ to satisfy the interpolation conditions $\Phi_N(\alpha_k)=\varphi(\alpha_k)$, $k=1, \ldots, N-1$. Since φ is strictly convex and $\varphi(0)=0$ we conclude that $\beta_k>0$, $k=1, \ldots, N-1$. Evidently the functions $\Phi_N(x)$ tend uniformly to $\varphi(x)$ in $[0, M]$ as N tends to infinity. Therefore $L_N(f):=\int_{-1}^{1}\Phi_N(|f'(x)|)dx$ tends to $L(f)$ as $N\to\infty$. Thus the inequality $L(f)\leq L(T_n)$ will be proved if we show that

$$\int_{-1}^{1}\varphi_\alpha(|f'(x)|)dx\leq\int_{-1}^{1}\varphi_\alpha(|T_n'(x)|)dx \tag{11}$$

for each $\alpha\in(0, M)$. But, from the definition of $\varphi_\alpha(x)$, $\int_{-1}^{1}\varphi_\alpha(|f'(x)|)dx =\int_{E(\alpha\,;\,f)}|f'(x)|\,dx$, where $E(\alpha\,;f):=\{x\in[-1, 1]:|f'(x)|\geq\alpha\}$. Clearly $E(\alpha\,;f)$ consists of non-overlapping intervals. Using Property 1, it is not difficult to see that

$$\int_{E(\alpha\,;\,f)}|f'(x)|\,dx<\int_{E(\alpha\,;\,T_n)}|T_n'(x)|\,dx,$$

which yields (11). Therefore $L_N(f)<L_N(T_n)$ for each natural N. In fact we have even more: there is a $\delta>0$ such that $L_N(f)<L_N(T_n)-\delta$ for each N Indeed,

$$\int_{-1}^{1}\varphi_0(|f'(x)|)dx<2n-1<2n=\int_{-1}^{1}\varphi_0(|T_n'(x)|)dx$$

for $f\neq T_n$. The last inequality remains true if we replace φ_0 by φ_α for each α from a sufficiently small neighbourhood $(0, \varepsilon_0)$ of 0. This assures the existence of the δ mentioned above. Then letting N tend to infinity we get (9). The theorem is proved.

Corollary 1. *Let n be an arbitrary natural number and $1<p<\infty$. Then $\|f'\|_{L_p[-1, 1]}\leq\|T_n'\|_{L_p[-1, 1]}\cdot\|f\|$ for each $f\in\pi_n$. The equality is attained if and only if $f=\pm T_n$.*

This is an extension of the classical A. A. Markov inequality which treats the case $p=\infty$. The assertion follows from Theorem 1 putting $\varphi(x)=x^p$.

Denote by $l(f)$ the arc-length of f in the interval $[-1, 1]$, i.e. $l(f)=\int_{-1}^{1}[1+f'^2(x)]^{1/2}dx$.

Corollary 2. *Let n be an arbitrary natural number and $M>0$. Then $l(f)\leq l(MT_n)$ for each $f\in\pi_n$ such that $\|f\|\leq M$. The equality is attained if and only if $f=\pm MT_n$.*

The assertion follows from the theorem in the case $\varphi(x)=(1+x^2)^{1/2}$.

Corollary 2 proves the known conjecture of Erdös [2] about the longest polynomial.

R e m a r k 1. Suppose that $f \in \pi_n$ and f has n distinct real zeros in $[-1, 1]$. Denote by $\{h_k\}_0^n$ the absolute values of the extremal values of f in $[-1, 1]$, i. e., $h_k = |f(t_k)|$, $k=0, \ldots, n$, where $t_0=-1$, $t_n=1$, $f'(t_i)=0$, $i=1, \ldots, n-1$. There is a theorem [3] (see also [1]), known as theorem of Davis, which gives a 1-1 correspondence between the system $\{h_k\}_0^n$ of extremums and the polynomials with real zeros. Precisely, given $\{h_k\}_0^n$, there exists a unique polynomial $f \in \pi_n$ and a unique system of points $-1=t_0<t_1<\ldots<t_n=1$ such that

$$f(t_k)=(-1)^{n-k} h_k, \quad k=0, \ldots, n.$$

Therefore the quantities $\{h_k\}_0^n$ determine the polynomial f uniquely. Thus the functional $L(f)$ can be considered as a function $L(h_0, \ldots, h_n)$ of $\{h_k\}_0^n$. One can prove that $L(h_0, \ldots, h_n)$ is a strictly increasing function with respect to h_k, $k=0, \ldots, n$, i. e. that $\partial L / \partial h_k > 0$, $k=0, \ldots, n$. Then, assuming that $f \in B(\pi_n)$, we get

$$L(f)=L(h_0, \ldots, h_n) \leqq L(1, \ldots, 1)=L(T_n)$$

with equality if and only if $h_0=\ldots=h_n=1$, i.e., iff $f=T_n$. This was the main original idea of our proof. Studying $\partial L / \partial h_k$ we came to the quantity $\sigma_k'(0)$ and consequently to the direct method based on the variation of $f(x)$ by $\varepsilon g_k(x)$.

P o s t s c r i p t. During the Conference on Constructive Function Theory in Varna, May — June, 1981 (after this paper was completed), J. Szabados called our attention to a paper of G. K. K r i s t i a n s e n (Some inequalities for algebraic and trigonometric polynomials. *J. London Math. Soc.*, **20**, 1979, No 2, 300-314) who treats a very similar problem. We were not able to check up the sketch of his complicated proof (especially the reasonings after Lemma 4). Our proof is more simple.

REFERENCES

1. B. D. B o j a n o v. A generalization of Chebyshev polynomials. *J. Approx. Theory*, **26** 1979, No 4, 293-300.
2. P. E r d ö s. An extremum-problem concerning trigonometric polynomials. *Acta Sci. Math. Szeged*, **9**, 1939, 113-115.
3. C. F i t z g e r a l d, L. L. S c h u m a k e r. A differential equation approach to interpolation at extremal points. *J. Analyse Math.*, **22**, 1969, 117-134.
4. Л. В. Т а й к о в. Одно обобщение неравенства С. Н. Бернштейна. *Труды Мат. инст. АН СССР*, **78**, 1965, 43-47.

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