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REPRESENTATION OF COMMUTANTS AND MULTIPLIERS CONNECTED WITH STURM-LIOUVILLE EXPANSIONS

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Summary. Convolutional representations of classes of commutants and multipliers closely connected with the generalized Sturm-Liouville expansions, i. e. with the expansions on the eigenfunctions and the associated functions of the spectral problem $y''-q(t)y=\lambda y$, $\alpha y(0)+\beta y'(0)=0$, $\chi(y)=0$, where χ is a continuous linear functional in $C^1[0,1]$, are obtained.

0. Preliminaries. With L^p, C^k, BV and AC are denoted the spaces $L^p[0,1]$, $C^k[0,1]$, the space of functions with bounded variation and the space of absolutely continuous functions in $[0,1]$. Let $BV^1=\{f\in AC: f'\in BV\}$, $BV_0^1=\{f\in BV^1: f(0)=0\}$ and let $AC^1=\{f\in C^1: f'\in AC\}$.

We consider the Sturm-Liouville operator $D=d^2/dt^2-q(t)$, with complex-valued $q\in L^1$ in its natural domain AC^1. Let $\chi_0(f)=\alpha f(0)+\beta f'(0)$, $|\alpha|+|\beta|\neq 0$ and let χ be a continuous linear functional in C^1. Let also $X_D=\{f\in AC^1: Df\in X,\ \chi_0(f)=\chi(f)=0\}$. In the present paper we try to solve by certain conditions two main problems:

1) For $X=L^1$, C or BV to describe the (X, X)-commutant of D with respect to X_D, i. e. to represent the set of all continuous linear operators $M: X\to X$, $M(X_D)\subset X_D$ commuting with D in X_D, i. e. $DMf=MDf$ for all $f\in X_D$.

2) For $X=L^1$, C or BV to describe the set of (X, X)-coefficient multipliers of the formal generalized Sturm-Liouville expansion $f\sim\sum_{n=1}^{\infty}P_nf$, when $\{P_n\}_{n=1}^{\infty}$ is a total projection system determining this expansion (for definition of P_n see Section 2), i. e. of these operators $M: X\to X$ satisfying $P_nMf=\mu_nP_nf$ for each $f\in X$. This is an expansion on the eigenfunctions and the associated functions of the problem $Dy=\lambda y$, $\chi_0(y)=\chi(y)=0$.

Our approach uses a convolution connected with this spectral problem has been introduced from I. Dimovski and the author in [1]. The basis of the considerations is the differentiation properties of this convolution, when $q(t)\equiv 0$, i. e. of the Dimovski's convolutions [2]. Section 1 is devoted to the differentiation properties and multipliers of Dimovski's convolutions. These results are generalized in Section 2 for arbitrary $q\in L^1$, where the problems 1) and 2) are solved for certain types of functionals $\chi \in C^{1*}$. In the paper the symbol $=_{\text{a. e.}}$ denotes equality almost everywhere.

1. Differentiation Properties and Multipliers of Dimovski Convolutions for d^2/dt^2.

Definition 1 ([3]). Let $M: X \to X$ be a linear operator in a linear space X. A bilinear, commutative and associative operation $f * g$ in X is said to be a convolution of M, iff

$$(1) \qquad M(f * g) = Mf * g = f * Mg \text{ for all } f, g \in X.$$

Every $M: X \to X$ satisfying (1) is called (X, X)-multiplier of $*$.

Let us consider the nonlocal boundary value problem $y'' - \lambda y = f$, $\chi_0(y) = \chi(y) = 0$. Without loss of generality we consider the model cases: (I) $\chi_0(y) = y'(0)$, (II) $\chi_0(y) = y(0)$. The resolvents defined by the previous problem corresponding to the cases (I) and (II) are denoted by R^1_λ and R^2_λ. They are resolvents of the operator d^2/dt^2 considered in the space $AC^1_\chi = \{f \in AC^1 : \chi_0(f) = \chi(f) = 0\}$ and are continuous operators in L^1, C, BV. Let us introduce the entire functions $E_1(\lambda) = \chi_t\{\operatorname{ch}\sqrt{\lambda}\, t\}$ and $E_2(\lambda) = \chi_t\{\operatorname{sh}\sqrt{\lambda}\, t/\sqrt{\lambda}\}$.

Definition 2. Dimovski convolutions for d^2/dt^2 are said to be the bilinear, commutative and associative operations

$$(2) \qquad f *_1 g = \chi(1) \int_0^t d\xi \int_0^\xi f(\xi - \tau) g(\tau) d\tau - \frac{1}{2} \chi_x \{ \int_0^x f \circ_1 g(\xi, t) d\xi \}$$

$$(3) \qquad f *_2 g = -\frac{1}{2} \chi_x \{ \int_0^x f \circ_2 g(\xi, t) d\xi \}, \text{ where } t \in [0, 1] \text{ and}$$

$$f \circ_1 g(x, t \overset{\text{def}}{=} \int_t^x f(x + t - u) g(u) du + \int_{-t}^x f_1(x - t - u) g_1(u) du,$$

$$f \circ_2 g(x, t) \overset{\text{def}}{=} \int_t^x f(x + t - u) g(u) du - \int_{-t}^x f_2(x - t - u) g_2(u) du.$$

Here $f_1(t) \overset{\text{def}}{=} f(|t|)$ and $f_2(t) \overset{\text{def}}{=} f(|t|) \operatorname{sgn} t$ denote the even and the odd extension of f in $[-1, 1]$, respectively.

By a direct calculation it can easily be verified that the resolvents R^i_λ, $i = 1, 2$, can be represented by the convolution $f *_i g$:

$$(4) \qquad R^i_\lambda f = \{ y_i(\lambda, t)/E_i(\lambda) \} *_i f, \; f \in C, \; i = 1, 2,$$

where $y_1(\lambda, t) = \operatorname{ch}\sqrt{\lambda}\, t$ and $y_2(\lambda, t) = \operatorname{sh}\sqrt{\lambda}\, t / \sqrt{\lambda}$ is the fundamental system of the equation $y'' - \lambda y = 0$.

Theorem 1. *The operation $f *_i g$, $i = 1, 2$, can be extented by (2), (3) as continuous bilinear commutative and associative operation in L^1 if in these formulas the Lebesgue continuation of the functional $\chi_x\{\int_0^x \cdot\} \in C^*$ is understood. Formula (4) holds for $f \in L^1$, too. The operation $f *_i g$, $i = 1, 2$, is a continuous bilinear operation $L^p \times L^1 \to L^p$, $L^p \times L^p \to L^p$, $1 \le p \le \infty$; $L^p \times L^q \to L^r$ for $1/p + 1/q = 1 + 1/r$, $1 \le p, q, r \le \infty$; $L^p \times L^q \to C$ for $1/p + 1/q = 1$, $1 \le p \le \infty$; $L^p \times L^p \to C$ for $2 \le p \le \infty$; $C \times C \to C$; $L^1 \times BV \to AC$; $L^1 \times AC \to AC$; $L^1 \times BV^1 \to AC^1$ for $i = 1$ and $L^1 \times BV^1_0 \to AC^1$ for $i = 2$.*

P r o o f. It is clear that $f \circ_i g(x, t) = f \circ g(x, t) + (-1)^{i+1} f \circ g(x, -t)$, where $f \circ g(x, t) = \int_t^x f(x+t-u)g(u)du$. Now all the propositions of the theorem follow from analogous properties of Berg-Dimovski convolution $N_x\{f \circ g(x, t)\}$ established from the author in [4], if N is taken to be the Lebesgue continuation of the functional $\chi_x\{\int_0^x f\}$.

R e m a r k. In the cases $L^1 \times BV \to AC$, $L^1 \times AC \to AC$ or in the cases $L^1 \times BV^1 \to AC^1$, $L^1 \times BV_0^1 \to AC^1$ the derivatives $(f *_i g)'$ or $(f *_i g)''$ can be calculated by differentiation formulas of Berg-Dimovski convolution ([4, (2.7) and (2.1)]). However, we formulate more special results we are needed:

In the next three theorems $\lambda \in \mathbf{C}$ is fixed with $E_i(\lambda) \neq 0$.

T h e o r e m 2. *a) If* $\chi \in C^*$, *then* $f *_i g$, $i=1, 2$, *is a continuous bilinear, operation* $L^1 \times L^1 \to AC$, $C \times C \to C^1$, $BV \times BV \to \{f \in AC^1 : f'' \in BV\}$ *and*

$$(f *_i g)' = -\frac{1}{2}\chi_x\{f \circ g(x, t)\} + \frac{(-1)^{i+1}}{2}\chi_x\{f_i \circ g_i(x, -t)\} \text{ for } f, g \in L^1.$$

Here under χ *we understand also the Lebesgue continuation of the primary* $\chi \in C^*$. *The operation* $f \tilde{*}_i g = (d^2/dt^2 - \lambda)(f *_i g)$, $i=1, 2$, *is a continuous convolution for* R_λ^i *is* BV *with unit element* $e_i(\lambda) = y_i(\lambda, t)/E_i(\lambda)$.

b) If $\chi(f) = \int_0^1 \gamma(u) f(u) du$ *with* $\gamma \in BV$, *then* $f *_i g$, $i=1, 2$, *is a continuous bilinear operation* $L^1 \times L^1 \to AC^1$, $C \times C \to C^2$ *and*

$$(f *_i g)'' = \frac{1}{2} F_x\{f \circ_i g(x, t)\} + \chi(g) f + \chi(f) g, \tag{5}$$

where F *denotes the Lebesgue continuation of the functional* $\int_0^1 f d\gamma - \gamma(1) f(1) + \gamma(0) f(0)$. *The operation* $f \tilde{*}_i g$ *is a continuous convolution for* R_λ^i *in* L^1 *and* C, *too, with the same unit element.*

The proof follows analogously using ([4, Theorems 2.6, 2.7]).

We note that in both cases a) and b) $\chi_0(f *_i g) = \chi(f *_i g) = 0$ hold for all $f, g \in BV$; $f, g \in L^1$ or $f, g \in C$ respectively.

T h e o r e m 3. *a) Let* $\chi \in C^*$. *Then an operator* M *is* (BV, BV)*-multiplier of* $f *_i g$, $i=1, 2$, *iff it admits a representation of the form*

$$Mf = (d^2/dt^2 - \lambda)(m *_i f) \text{ with } m \in BV, f \in BV. \tag{6}$$

b) Let $\chi(f) = \int_0^1 \gamma(u) f(u) du$ *with* $\gamma \in BV$. *An operator is a* (L^1, L^1) *or* (C, C)*-multiplier of* $f *_i g$, $i=1, 2$, *iff it admits a representation of the form* (6) *with* $m \in L^1$, $f \in L^1$ *or with* $m \in C$, $f \in C$, *respectively. Now* (6) *can be expressed by* (5).

Multipliers in the cases when χ is not so "smooth" can be obtained in the form (6), too, but it is necessary to characterize more precisely the class of the representation functions m. This can be done using necessary conditions for differentiation of $f *_i g$, $i=1, 2$, as it is made in the next theorem:

T h e o r e m 4. *Let* $\chi_0(y) = y'(0)$, $\chi(y) = y'(1) + \Phi(y)$ *with* $\Phi \in C^*$.

a) If $g \in L^1$ *and if* $f *_1 g \in AC^1$ *for all* $f \in L^1$, *then* $g =_{a.e.} \tilde{g} \in BV^1$.

b) An operator M *is a* (L^1, L^1)*-multiplier of* $f *_1 g$ *iff it admits a representation of the form* (9) *with* $m \in BV^1$, $f \in L^1$.

The proof is based on four lemmas. Let $L_e^1 = \{f \in L^1 : f(t) =_{a.e.} f(-t)\}$.

Lemma 1. *Let* $M:L^1_e \to L^1[-1,1]$ *be a* (L^1_e, L^1)*-multiplier of the finite Fourier transform* $F_n(f)=\int_{-1}^{1} e^{in\pi\tau} f(\tau)d\tau$, $f\in L^1[-1,1]$, *i.e.* $F_n(Mf) = \nu_n F_n(f)$ *for all* $f\in L^1_e$. *Then there is a Borel measure* μ *such that* $Mf = f * d\mu = \int_{-1}^{1} f(t-\tau)d\mu(\tau)$ *for all* $f\in L^1_e$.

Proof. First from the closed graph theorem we get that M is continuous. If $h=M(f*g)-f*Mg$ then $F_n(h)=0$ for all $n=0, \pm1, \pm2, \dots$, hence $h=0$ and (1) holds for all $f, g\in L^1_e$. Now using that the usual approximate identities (e. g. Feier kernels) belong to L^1_e we reduce the proof to a well-known procedure ([5, p. 269]).

Lemma 2. *Let* $M:L^1\to C$ *be a linear operator and let* $\|Mf\|_{L^1} \leqq k\|f\|_{L^1}$ *for each* $f\in L^1$. *Then there exists* k_1 *such that* $\|Mf\|_C \leqq k_1\|f\|_{L^1}$ *for all* $f\in L^1$.

The proposition follows from the closed graph theorem. We note that the lemma holds true when $M:L^1_e \to C[-1,1]$ since L^1_e is a closed subspace of $L^1[-1,1]$.

Lemma 3. *Let* $g\in L^1$ *be such that* $f *_i g\in AC^1$ *for all* $f\in L^1$. *Then* $\chi_0(f *_i g)=\chi(f *_i g)=0$ *for all* $f\in L^1$.

The proof uses an approximation, Lemma 2 and (4).

Lemma 4. *a) Let* $g\in L^1$ *and let* $f \circ_1 g(1,t)\in AC$ *for all* $f\in L^1$ *then* $g=_{\text{a.e.}} \tilde{g}\in BV$.

b) Let $g\in L^1$ *and let* $f\circ_1 g(1,t)\in AC^1$ *for all* $f\in L^1$ *then* $g=_{\text{a.e.}} \tilde{g}\in BV^1$.

Proof. In the case a) $f\circ_1 g_1(1,t)\in AC[-1,1]$ for all $f\in L^1_e$, $t\in[-1,1]$. From Lemma 3 we have $f\circ_1 g(1,t)'|_{t=0}=0$. Since its proof holds true for the operation $f\circ_1 g(1,t)$ when $f, g\in L^1_e$ and $t\in[-1,1]$, we get $f\circ_1 g_1(1,t)'|_{t=0} = f\circ_1 g_1(1,t)'|_{t=1}=0$ for all $f\in L^1_e$. That means $f\circ_1 g_1(1,t)\in AC^1[-1,1]$ in the case b) for all $f\in L^1_e$, i. e. the operators $M_1 f=f\circ_1 g_1(1,t)'$ and $M_2 f = f\circ_1 g_1(1,t)''$ are defined in the whole L^1_e. We use the fact that $f\circ_1 g(1,t)$ for $f, g\in L^1_e$ is a convolution for the finite cosine Fourier transform $C_n(f) = \int_{-1}^{1} f(\tau)\cos n\pi\tau d\tau$, with the property $C_n[f\circ_1 g(1,t)]=(-1)^n C_n(f)C_n(g)$. This can be verified first for the functions $f=\cos n\pi t$, $g=\cos m\pi t$, $n, m=0, 1, \dots$, and by approximation with their linear combinations to establish this equality in L^1_e. Now using that $F_n(f)=C_n(f)$ for $f\in L^1_e$ and Lemma 3 we get that M_1 and M_2 are (L^1_e, L^1)-multipliers of $F_n(f)$. Then by Lemma 1 there exist measures μ_1, μ_2 such that $M_i f=f*d\mu_i$ for $f\in L^1_e$. By elementary calculations we get $M_i(t^2)=_{\text{a.e.}} a_i t^2+b_i t+c_i+d_i\int_0^{t-1}\mu_i(\tau)d\tau$ for $t\in[0,1]$ where $a_i, b_i, c_i, d_i\in \mathbf{C}$, $d_i\neq 0$. In the case a) we get $M_1(t^2)=_{\text{a.e.}} -2\int_0^t g+\alpha t$ with $\alpha\in\mathbf{C}$, $t\in[0,1]$, hence the right parts coincide everywhere. Therefore $g=_{\text{a.e.}}\tilde{g}\in BV$. In the case b) we get $M_2(t^2)=_{\text{a.e.}} -2g+\alpha$, hence $g=_{\text{a.e.}}\tilde{g}\in BV^1$.

Proof of Theorem 4. a) Now $f *_1 g=1/2 f\circ_1 g(1,t)+f *_\Phi g$, where $*_\Phi$ is of the form (2) with χ replaced by $\Phi\in C^*$. Then Theorem 2 a) shows that $f *_\Phi g\in AC$ for all $f\in L^1$, hence $f\circ_1 g(1,t)\in AC$ for all $f\in L^1$ and by Lemma 4 a) we have $g=_{\text{a.e.}} h\in BV$. Now Theorems 2 a) and 1 b) show that $f *_\Phi g\in AC^1$ for all $f\in L^1$, hence $f\circ_1 g(1,t)\in AC^1$ for all $f\in L^1$ and by Lemma 4 b) we get $g=_{\text{a.e.}} \tilde{g}\in BV^1$. b) Let M be a (L^1, L^1)-multipler of $f *_1 g$.

Then by (4) $M\{\mathrm{ch}\sqrt{\lambda}\,t/E_1(\lambda)\} *_1 f=\{\mathrm{ch}\sqrt{\lambda}\,t/E_1(\lambda)\} *_1 Mf=R_\lambda^1 Mf \in AC^1$ holds for each $f\in L^1$, hence $m\overset{\text{def}}{=}M\{\mathrm{ch}\sqrt{\lambda}\,t/E_1(\lambda)\}\in BV^1$ and (6) holds. Conversely, if $m\in BV$, then from (4) and Theorem 1 it follows that $Mf \overset{\text{def}}{=}(d^2/dt^2-\lambda)(m *_1 f)$ is a (L^1, L^1)-multiplier of $f *_1 g$.

2. Multipliers and Commutants Connected with Generalized Sturm-Liouville Expansions. Let $D=d^2/dt^2-q(t)$, $q\in L^1$ be the Sturm-Liouville operator considered in AC^1. Božinov and Dimovski [1] have introduced a convolution $f * g$ in C for the operator D, when $q\in C$, representing the resolvent R_λ of D considered on the space $AC_\chi^1=\{f\in AC^1:\chi_0(f)=\chi(f)=0\}$, i. e. $y=R_\lambda f$, is a solution of the problem $Dy-\lambda y=f$, $\chi_0(y)=\alpha y(0)+\beta y'(0)=0$, $\chi(y)=0$, where $\chi\in C^{1*}$. Let $\{y_1(\lambda, t), y_2(\lambda, t)\}$ be the fundamental system of the equation $Dy=\lambda y$, i. e. $y_1(\lambda, 0)=1$, $y_1'(\lambda, 0)=0$ and $y_2(\lambda, 0)=0$, $y_2'(\lambda, 0)=1$ hold, and let $y(\lambda, t)\overset{\text{def}}{=}\beta y_1(\lambda, t)-\alpha y_2(\lambda, t)$. Let us introduce also the entire function $E(\lambda)=\chi_t\{y(\lambda, t)\}$. Then the resolvent has the representation

$$R_\lambda f=\{y(\lambda, t)/E(\lambda)\} * f \text{ for } f\in C. \tag{7}$$

The convolution $f * g$ has the form $f * g=T_i^{-1}(T_i f *_i T_i g)$, where $T_i f=f(t)+\int_0^t K_i(t, \tau)f(\tau)d\tau$ is Delsarte-Povzner transmutation operator with suitable chosen kernel $K_i(t, \tau)$ and where $*_i$, $i=1, 2$, are of the form (2) or (3), where χ is replaced by $\chi(T_i^{-1}f)$. We take $i=1$ if $\beta\neq 0$ and $i=2$ if $\beta=0$. The operator T_i is a linear continuous isomorphism mapping L^1, L^p, BV, C, AC, BV^1, AC^1 onto itself. It can be proved that when $q\in L^1$

$$T_i D f=d^2/dt^2\, T_i f \tag{8}$$

holds for each $f\in AC^1$ with $\chi_0(f)=0$. The equality (7) follows from (8) since from it follows $R_\lambda f=T_i^{-1}R_\lambda^i T_i f$ for each $f\in L^1$. Thus we can obtain (7) from (4) using T_i and that $y(\lambda, t)=\text{const.}\, T_i(y_i(\lambda, t))$.

Theorem 5. *The operation $f * g$ extends as continuous bilinear commutative and associative operation in L^1 such that* (7) *holds for $f\in L^1$, too. It is a continuous bilinear operation $L^p\times L^1\to L^p$, $L^p\times L^p\to L^p$ for $1\leq p\leq\infty$; $L^p\times L^q\to L^r$ for $1/p+1/q=1+1/r$, $1\leq p, q, r\leq\infty$; $L^p\times L^q\to C$ for $1/p+1/q=1$, $1\leq p\leq\infty$; $L^p\times L^p\to C$ for $2\leq p\leq\infty$; $C\times C\to C$; $L^1\times BV\to AC$; $L^1\times AC\to AC$; $L^1\times BV^1\to AC^1$ if $\beta\neq 0$ and $L^1\times BV_0^1\to AC^1$ if $\beta=0$.*

The theorem follows from Theorem 1 using T_i.

Theorem 6. *Let $X=L^1$, C* or *BV. Then an operator $M:X\to X$ belongs to the (X, X)-commutant of D relative to X_D (for definition see Section 0) iff M is a multiplier of $f * g$.*

Proof. It is easy to prove that M belongs to the commutant iff $MR_\lambda f=R_\lambda Mf$ for each $f\in X$ and each $\lambda\in\mathbf{C}$ with $E(\lambda)\neq 0$. Using (7) we get $M(y(\lambda^2, t) * g)=y(\lambda^2, t) * Mg$ for each $g\in L^1$ and each $\lambda\in\mathbf{C}$ with $E(\lambda^2)\neq 0$ but from the analycity of both the parts with respect to λ it follows that this equality holds for all $\lambda\in\mathbf{C}$. Since $y(\lambda^2, t)=T_1\{\mathrm{ch}\,\lambda t\}$, when $i=1$ ($\beta\neq 0$) and $y(\lambda^2, t)=T_2\{\mathrm{sh}\,\lambda t/\lambda\}$, when $i=2$ ($\beta=0$), and since every function of X can be approximated with linear combinations of the classes $\{\mathrm{ch}\,\lambda t\}_{\lambda\in\mathbf{C}}$ and $\{\mathrm{sh}\,\lambda t/\lambda\}_{\lambda\in\mathbf{C}}$ with respect to $\|\,.\,\|_{L^1}$, then it follows that every $f\in X$ can be approximated with linear combinations of the functions $y(\lambda^2, t)$, $\lambda \in \mathbf{C}$, with respect to $\|.\|_{L^1}$. But if $f_n, f \in X$, $f_n \xrightarrow{L^1} f$ then for arbitrary fixed $g \in X = L^1, C$ or BV from Theorem 5 it follows that $f_n * g \xrightarrow{X} f * g$ and by approximation we get $M(f * g) = f * Mg$ for all $f \in X$. Conversely, iff (1) holds, then from (7) we get $MR_\lambda f = R_\lambda Mf$ for all $f \in X$ and each λ, $E(\lambda) \neq 0$.

Theorem 7. *a) Let $\chi \in C^*$. Then an operator M is a (BV, BV)-multiplier of $f * g$ iff it admits a representation of the form*

$$Mf = (D - \lambda)(m * f) \text{ with } m \in BV,\ f \in BV. \tag{9}$$

b) Let $\chi(f) = \int_0^1 \gamma(u) f(u) du$ with $\gamma \in BV$. Then an operator M is (L^1, L^1)- or (C, C)-multiplier of $f * g$ iff it admits a representation of the form (9) with $m \in L^1$, $f \in L^1$ or with $m \in C$, $f \in C$.

c) Let $\chi_0(f) = y'(0) + hy(0)$ and let $\chi(y) = y'(1) + \Phi(y)$ with $\Phi \in C^*$. Then an operator M is a (L^1, L^1)-multiplier of $f * g$ iff it admits a representation of the form (9) with $m \in BV^1$, $f \in L^1$.

The theorem follows immediately from Theorems 3, 4 using the transmutation operator T_i and the Equality (8).

Now we shall do some applications for the generalized Sturm-Liouville expansion determined by the spectral problem $Dy = \lambda y$, $\alpha y(0) + \beta y'(0) = 0$, $\chi(y) = 0$, where $\chi \in C^{1*}$ is a functional of the types in Theorem 7, when $X = L^1$, BV or C.

Lemma 5. *Let $\chi \in C^{1*}$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be all the zeros of $E(\lambda)$ connected with the spectral problem. Let Γ_n be a contour enclosing only λ_n of these zeros in its inside. Then the projection $P_n = -\frac{1}{2\pi i} \int_{\Gamma_n} R_\lambda d\lambda$ (mapping X onto the root subspace H_n corresponding to λ_n) has the multiplier form $P_n f = f * \varphi_n$, where $\varphi_n(t) = -\frac{1}{2\pi i} \int_{\Gamma_n} \frac{y(\lambda, t)}{E(\lambda)} d\lambda \in H_n$ has the properties $\varphi_n * \varphi_m = 0$ for $n \neq m$ and $\varphi_n * \varphi_n = \varphi_n$.*

The lemma follows from (7). See also [6].

There are many research works studying when the projection system $\{P_n\}_{n=1}^{\infty}$ is total in X, i. e. when $P_n f = 0$ for all $n = 1, 2, \ldots$ implies $f = 0$. This is surely true ([6]), when the system S consisting of all eigenfunctions and associated functions of the spectral problem is complete in X, i. e. when its linear span $\langle S \rangle$ is dense in X. We shall not discuss this here, but we remark only that to this class belongs the classical Sturm-Liouville problem, when $\chi_0(y) = y'(0) + hy(0)$, $\chi(y) = y'(1) + ky(1)$ (for the case when q is complex valued see e. g. [7]). A review can be found in [8].

Theorem 8. *Let us suppose that $\{P_n\}_{n=1}^{\infty}$ is a total system in $X = L^1$, BV or C.*

a) Let χ be as in Theorem 7 a), b) or c). An operator M in X is a (X, X)-coefficient multiplier of Sturm-Liouville expansion $f \sim \sum_{n=1}^{\infty} P_n f$ iff M admits a representation of the form (9) with $m \in BV$, $f \in BV$ in the case a), with $m \in L^1$, $f \in L^1$ or with $m \in C$, $f \in C$ in the case b) and with $m \in BV^1$, $f \in L^1$ in the case c), but in the three cases with $m \sim \sum_{n=1}^{\infty} \mu_n R_\lambda \varphi_n$, i. e. with $m \sim \sum_{n=1}^{\infty} \mu_n P_n \{y(\lambda, t)/E(\lambda)\}$.

b) If $E(\lambda)$ has simple zeroso, nly then the set of (X, X)-coefficient multipliers coincides with (X, X)-commutant of D relative to X_D, i. e. with the set of all (X, X)-multipliers of $f * g$. If $\chi \in C^{1*}$ is as in Theorem 7 a), b) or c) then by (9) *Theorem 7 gives a complete description of the set of (X, X)-coefficient multipliers with m of the form as in cases a), b), c) of Theorem 7.*

The theorem follows from a general result of the author [6].

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