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A REFINEMENT OF THE KOLMOGOROV-CRITERION

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Summary. A refinement of the Kolmogorov-criterion is given. This refinement is used to give a characterization of those functions in $C_0(T)$, which have a strongly unique best approximation from a finite dimensional subspace in $C_0(T)$. Extensions of the refined criterion to normed linear spaces and to optimization problems are discussed.

1. Introduction. This paper deals with a refinement of the well-known Kolmogorov-criterion. We begin here by recalling some definitions and earlier results. In Section 2 we prove the refined Kolmogorov-criterion, in Section 3 we apply this refinement to prove a strong uniqueness theorem, and in Section 4 we show how this criterion can be extended to normed linear spaces. Finally, we discuss a generalization to optimization problems.

Let T be a locally compact Hausdorff-space and let $C_0(T)$ denote the real vector space of all continuous functions $x: T\to \boldsymbol{R}$ vanishing at infinity, i. e. for each $\varepsilon>0$ the set

$$\{t\in T \mid |x(t)| \geqq \varepsilon\}$$

is compact. We assume, that $C_0(T)$ is endowed with the norm $\|x\| = \sup\{|x(t)| \in \boldsymbol{R} \mid t\in T\}$. Let $V := \operatorname{span}(v_1, v_2, \ldots, v_N)$ be an N-dimensiona linear subspace of $C_0(T)$ and let x be an element in $C_0(T)$. An element $v_0\in V$ is said to be a best approximation to x in V, if $\|x-v_0\| = \inf\{\|x-v\| : v\in V\} =: E_V(x)$. We denote by $P_V(x)$ the set of all best approximations to x in V.

For each x in $C_0(T)$ we denote by M_x the following set

$$M_x := \{t\in T \mid |x(t)| = \|x\|\}.$$

A best approximation to x in V can be characterized by the following property:

Theorem. *An element v_0 is a best approximation to x from V if and only if*

$$\forall_{v\in V} \min_{t\in M_{x-v_0}} (x(t)-v_0(t))v(t) \leqq 0.$$

This criterion for a best approximation is called the Kolmogorov-criterion.

For the refinement of the criterion, we introduce *extremal signatures*. A *signature* ε on T is a continuous mapping of a closed subset of T into $\{-1, 1\}$. A signature ε is called extremal with respect to the linear subspace V if

(E) $$\forall_{v \in V} \min_{t \in \mathrm{DOM}(\varepsilon)} \varepsilon(t) \cdot v(t) \leqq 0.$$

An extremal signature ε is called *primitive* if for any closed subset $F \subset \mathrm{DOM}(\varepsilon)$ with $F \neq \mathrm{DOM}(\varepsilon)$ the restriction $\varepsilon_{|F}$ is not extremal. For $x \neq 0$, there is a natural signature ε_x defined by $\varepsilon_x(t) \cdot x(t) = \| x \|$. With these definitions the Kolmogorov-criterion reads:

An element v_0 in V is a best approximation to x in V if and only if the signature ε_{x-v_0} is extremal with respect to V.

For each t in T, we define the vector

$$\mathfrak{V}(t) := \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_N(t) \end{pmatrix} \in \boldsymbol{R}^N.$$

Further, we denote by $\langle \cdot , \cdot \rangle$ the usual inner product and by $\| \cdot \|_2$ the Euclidean norm in $\boldsymbol{R}^N$. The condition (E) is equivalent to

$$0 \in \mathrm{con}\,(\{ \varepsilon(t) \cdot \mathfrak{V}(t) \in \boldsymbol{R}^N \mid t \in \mathrm{DOM}(\varepsilon)\}),$$

where $\mathrm{con}(\cdot)$ denotes the convex hull of a set. By the definition of the convex hull, one has $\sum_{\mu=0}^{m} \rho_\mu \varepsilon(t_\mu) \mathfrak{V}(t_\mu) = 0$ with $t_0, t_1, \ldots, t_m \in \mathrm{DOM}(\varepsilon)$, $\rho_0, \rho_1, \ldots, \rho_m \geqq 0$, and $\sum_{\mu=0}^{m} \rho_\mu = 1$. If ε is a primitive extremal signature, then $\mathrm{DOM}(\varepsilon)$ is finite and for each subset $F \subset \mathrm{DOM}(\varepsilon)$ with $F \neq \mathrm{DOM}(\varepsilon)$ the elements $\mathfrak{V}(t) \in \boldsymbol{R}^N$, $t \in F$, are linearly independent. Consequently, for a primitive extremal signature ε, one has $\sum_{t \in \mathrm{DOM}(\varepsilon)} \rho(t) \cdot \varepsilon(t) \mathfrak{V}(t) = 0$ with $\rho(t) > 0$ for each $t \in \mathrm{DOM}(\varepsilon)$.

2. The Refined Kolmogorov-Criterion. For each primitive extremal signature ε we introduce the linear subspace

$$V(\varepsilon) := \{ b \in \boldsymbol{R}^N \mid \forall_{t \in \mathrm{DOM}(\varepsilon)} \langle b, \mathfrak{V}(t) \rangle = 0 \}.$$

For any element $v = \sum_{\nu=1}^{N} b_\nu v_\nu = \langle b, \mathfrak{V} \rangle$ in V, we denote by $\theta_v(\varepsilon)$ the angle between b in $\boldsymbol{R}^N$ and the linear subspace $V(\varepsilon) \subset \boldsymbol{R}^N$. Now we prove the following refinement of the criterion of Kolmogorov:

Theorem 1. *Let v_0 be a best approximation to $x \in C_0(T) \setminus V$ from V and let ε be a primitive extremal signature contained in ε_{x-v_0}. Then, there exists a real number $K_1 > 0$ such that*

$$\forall_{v \in V} \min_{t \in M_{x-v_0}} (x(t) - v_0(t)) \cdot v(t) \leqq -K_1 \cdot \| v \| \, \theta_\varepsilon^2(v).$$

Proof. Since ε is a primitive extremal signature contained in ε_{x-v_0} there exist points $t_0, t_1, \ldots, t_m \in M_{x-v_0}$ such that $\{t_0, t_1, \ldots, t_m\} = \mathrm{DOM}(\varepsilon)$. Further, there exist real numbers $\rho_0, \rho_1, \ldots, \rho_m > 0$ with $\sum_{\mu=0}^{m} \rho_\mu = 1$ such that

(*) $$\sum_{\mu=0}^{m} \rho_\mu \varepsilon(t_\mu) \mathfrak{B}(t_\mu)=0.$$

For each b in $V(\varepsilon)$ one has $\langle b, \mathfrak{B}(t_\mu)\rangle=0$, $\mu=0, 1, \ldots, m$, hence span $(\mathfrak{B}(t_0), \mathfrak{B}(t_1), \ldots, \mathfrak{B}(t_m))=V(\varepsilon)^\perp$.

Now let b be any vector in $V(\varepsilon)^\perp$ with $\| b \|_2=1$. Then the numbers $\langle b, \mathfrak{B}(t_\mu)\rangle$ cannot be all zero. Since in the relation (*) $\rho_\mu>0$, we infer that at least one of $\varepsilon(t_\mu)\langle b, \mathfrak{B}(t_\mu)\rangle$ is negative. Consequently, the expression $\min_\mu \varepsilon(t_\mu) \langle b, \mathfrak{B}(t_\mu)\rangle$ is a negative function of b. Hence the number

$$\alpha:= \max_{\| b \|_2=1} \min_\mu \varepsilon(t_\mu) \langle b, \mathfrak{B}(t_\mu)\rangle$$

is negative. For an arbitrary vector b in $V(\varepsilon)^\perp$ one has $\min_\mu \varepsilon(t_\mu) \langle b, \mathfrak{B}(t_\mu)\rangle \leqq \alpha \| b \|_2$.

Now let b be an arbitrary vector in $\boldsymbol{R}^N$ and denote by $T_\varepsilon(b)$ the orthogonal projection of b on $V(\varepsilon)$. Then $b-T_\varepsilon(b)$ is contained in $V(\varepsilon)^\perp$. Now let $v=\sum_{\nu=1}^N b_\nu v_\nu=\langle b, \mathfrak{B}\rangle$ be an arbitrary element in V. Then we have the estimate

$$\begin{aligned}\min_\mu \varepsilon(t_\mu) v(t_\mu) &= \min_\mu \varepsilon(t_\mu) \langle b, \mathfrak{B}(t_\mu)\rangle = \min_\mu \varepsilon(t_\mu) \langle b-T_\varepsilon(b), \mathfrak{B}(t_\mu)\rangle \\ &\leqq \alpha \| b-T_\varepsilon(b)\|_2 \leqq \alpha\{\| b \|_2-\| T_\varepsilon(b)\|_2\} \\ &= \alpha \| b \|_2 (1-\cos\theta_\varepsilon(v)) \leqq -K_0 \cdot \| v \| \theta_\varepsilon^2,\end{aligned}$$

with a suitable real number $K_0>0$. Since

$$\varepsilon(t_\mu)=\operatorname{sign}(x(t_\mu)-v_0(t_\mu)), \quad \mu=0, 1, \ldots, m,$$

and since $\mathrm{DOM}(\varepsilon)\subset M_{x-v_0}$, it follows that

$$\min_{t \in M_{x-v_0}} (x(t)-v_0(t))\, v(t) \leqq -K_1 \cdot \| v \| \theta_\varepsilon^2,$$

with a suitable real number $K_1>0$. □

A simple consequence of Theorem 1 is the

Corollary 2. *Let v_0 be a best approximation to $x \in C_0(T)\setminus V$ from V and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ be primitive extremal signatures contained in ε_{x-v_0}. Then, there exists a real number $K_2>0$ such that*

$$\forall_{v \in V} \min_{t \in M_{x-v_0}} (x(t)-v_0(t))\cdot v(t) \leqq -K_2 \cdot \| v \| \cdot \max_\varkappa \theta_v^2(\varepsilon_\varkappa).$$

3. Strong Uniqueness. An element v_0 in V is called a strongly unique best approximation of the function x in $C_0(T)$ from V, if there exists a real number $K_3>0$ such that

$$\forall_{v \in V} \| x-v \| \geqq \| x-v_0 \| + K_3 \| v-v_0\|.$$

In general a unique best approximation is not a strongly unique best approximation. To characterize those functions x in $C_0(T)$, which have a strongly unique best approximation, we introduce the set

$$\Gamma_{x-v_0} := \cup \{\mathrm{DOM}(\varepsilon) \subset T \mid \varepsilon \subset \varepsilon_{x-v_0} \text{ and } \varepsilon \text{ primitive extremal}\},$$

where v_0 is a best approximation to x from V. Then we have the following

Theorem 3. *An element v_0 in V is a strongly unique best approximation to x in $C_0(T)$ from V if and only if there exist points $t_1, t_2, \ldots, t_N \in \Gamma_{x-v_0}$ such that* $\det(v_\nu(t_\mu)) \neq 0$.

Proof. (1) Assume that there exist points $t_1, t_2, \ldots, t_N$ in Γ_{x-v_0} such that $\det(v_\nu(t_\mu)) \neq 0$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ be primitive extremal signatures such that $\varepsilon_\varkappa \subset \varepsilon_{x-v_0}$, $\varkappa = 1, 2, \ldots, k$, and $\{t_1, t_2, \ldots, t_N\} \subset \bigcup_{\varkappa=1}^{k} \mathrm{DOM}(\varepsilon_\varkappa) =: \Gamma_1$. In view of Corollary 2, we have

$$\forall_{v \in V} \min_{t \in M_{x-v_0}} (x(t) - v_0(t)) v(t) \leq -K_2 \cdot \| v \| \cdot \max_{\varkappa} \theta_v^2(\varepsilon_\varkappa).$$

Since $\det(v_\nu(t_\mu)) \neq 0$, it follows that $\bigcap_{\varkappa=1}^{k} V(\varepsilon^\varkappa) = \{0\}$. Consequently, for each $v = \langle b, \mathfrak{B} \rangle$ in V there exists an index $\varkappa \in \{1, 2, \ldots, k\}$ such that $b \notin V(\varepsilon_\varkappa)$. Hence $\theta_v(\varepsilon_\varkappa) \neq 0$ and thus there exists a point $t \in M_{x-v_0}$ such that $(x(t) - v_0(t)) \cdot v(t) \leq -K_2 \| v \| \theta_v^2(\varepsilon_\varkappa) < 0$. Hence, the expression $J(v) := \min_{t \in M_{x-V_0}} (x(t) - v_0(t)) \cdot v(t)$ is a negative continuous function of v in $V \setminus \{0\}$. Then we have $\alpha := \max_{\|v\|=1} J(v) < 0$, which implies

$$\forall_{v \in V} \min_{t \in M_{x-v_0}} (x(t) - v_0(t)) \cdot v(t) \leq \alpha \cdot \| v \|.$$

For each $t \in M_{x-v_0}$ one has the estimate

$$\| x - v \| \geq \varepsilon_{x-v_0}(t)(x(t) - v(t)) = \varepsilon_{x-v_0}(t)(x(t) - v_0(t)) + \varepsilon_{x-v_0}(t)(v_0(t) - v(t)),$$

which implies

$$\| x - v \| \geq \| x - v_0 \| - \frac{\alpha}{\| x - v_0 \|} \cdot \| v - v_0 \| = \| x - v_0 \| + K_3 \cdot \| v - v_0 \|.$$

(2) Assume now that $v_0 \in V$ is a strongly unique best approximation to x. Then, by a result of Bartelt and McLaughlin [1],

$$0 \in \operatorname{int} \operatorname{con}(\{\varepsilon_{x-v_0}(t) \mathfrak{B}(t) \in \boldsymbol{R}^N \mid t \in M_{x-v_0}\}).$$

By a theorem of Steinitz (compare Danzer, Grünbaum and Klee [4]) there exist $k \leq 2N$ points $t_1, t_2, \ldots, t_k$ in M_{x-v_0} such that

$$0 \in \operatorname{int} \operatorname{con}(\{\varepsilon_{x-v_0}(t_\varkappa) \cdot \mathfrak{B}(t_\varkappa) \in \boldsymbol{R}^N \mid \varkappa = 1, 2, \ldots, k\}).$$

Now let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ denote the primitive extremal signatures with $\mathrm{DOM}(\varepsilon_\lambda) \subset \{t_1, t_2, \ldots, t_k\}$ and $\varepsilon_\lambda \subset \varepsilon_{x-v_0}$, $\lambda = 1, 2, \ldots, l$. If there exist only $n < N$ points $\tau_1, \tau_2, \ldots, \tau_n$ in $\bigcup_{\lambda=1}^{l} \mathrm{DOM}(\varepsilon_\lambda)$ such that $\mathfrak{B}(\tau_1), \mathfrak{B}(\tau_2), \ldots, \mathfrak{B}(\tau_n)$ are linearly independent, then

$$W := \operatorname{span}(\mathfrak{B}(\tau_1), \mathfrak{B}(\tau_2), \ldots, \mathfrak{B}(\tau_n))$$

is different from $\boldsymbol{R}^N$. Since

$$0 \in \operatorname{int} \operatorname{con}(\{\varepsilon_{x-v_0}(t_\varkappa) \mathfrak{B}(t_\varkappa) \in \boldsymbol{R}^N \mid \varkappa = 1, 2, \ldots, k\}),$$

there exists a point $\bar{t} \in \{t_1, t_2, \ldots, t_k\}$ such that $\varepsilon_{x-v_0}(\bar{t}) \mathfrak{B}(\bar{t}) \notin W$. Then there exists a line through 0 and $\varepsilon_{x-v_0}(\bar{t}) \mathfrak{B}(\bar{t})$ which intersects the boundary of

$$\operatorname{con}(\{\varepsilon_{x-v_0}(t_\varkappa) \mathfrak{B}(t_\varkappa) \in \boldsymbol{R}^N \mid \varkappa = 1, 2, \ldots, k\})$$

in a point w. This point is a convex combination $w=\Sigma_{\mu=1}^{m}\rho_\mu \varepsilon_{x-v_0}(s_\mu)\mathfrak{V}(s_\mu)$ with $s_1, s_2, \ldots, s_m$ in $\{t_1, t_2, \ldots, t_k\}$. We can assume that m is as small as possible.

Then

$$0 \notin \operatorname{con}(\{\varepsilon_{x-v_0}(s_\mu)\mathfrak{V}(s_\mu) \in \boldsymbol{R}^N \mid \mu=1, 2, \ldots, m\}).$$

The restriction of ε_{x-v_0} to the set $\{\bar{t}, s_1, s_2, \ldots, s_m\}$ defines an extremal signature ε_{l+1}, which contains a primitive extremal signature ε_{l+1} with $\bar{t} \in \mathrm{DOM}(\varepsilon_{l+1})$, which is a contradiction. Consequently, we have $n=N$. □.

4. Extension to Normed Linear Spaces. The refined Kolmogorov-criterion and the strong uniqueness theorem can be extended to normed linear spaces in the usual way. We first state some necessary definitions. Let X be a real normed linear space and $V:=\operatorname{span}(v_1, v_2, \ldots, v_N)$ be an N-dimensional subspace of X. To each element x in X we associate the set $P_V(x):=\{v_0 \in V \mid \|x-v_0\|=\inf_{v\in V}\|x-v\|\}$ which is called the set of best approximations for x by elements of V. Further we define $\mathfrak{S}_{X^*}:=\operatorname{ep}(\{x^* \in X^* \mid \|x^*\| \leq 1\})$, where $\operatorname{ep}(A)$ denotes the set of extreme points of a set A. For x in X, we set

$$\mathfrak{S}_x:=\{x^* \in \mathfrak{S}_{X^*} \mid \|x\|=x^*(x)\}.$$

We use the term σ_{ep}-topology to denote the restriction of the weak $\sigma(X^*, X)$-topology on the set $\mathfrak{S}_{X^*}$.

A σ_{ep}-closed subset $\mathfrak{S}$ of $\mathfrak{S}_{X^*}$ is called a signature if there exists an element x in $X\setminus\{0\}$ such that $\mathfrak{S}\subset\mathfrak{S}_x$. A signature $\mathfrak{S}$ is called extremal with respect to the linear subspace V if

$$\forall_{v \in V} \min_{x^* \in \mathfrak{S}} x^*(v) \leq 0.$$

An extremal signature $\mathfrak{S}$ is called primitive if any σ_{ep}-closed subset $\mathfrak{S}' \subset \mathfrak{S}$ with $\mathfrak{S}' \neq \mathfrak{S}$ is not extremal. A primitive extremal signature is a finite set. For each x in $X\setminus\{0\}$ there exists a natural signature, namely $\mathfrak{S}_x$. For each x^* in $\mathfrak{S}_{X^*}$ we define the vector

$$\mathfrak{V}(x^*):=\begin{pmatrix} x^*(v_1) \\ x^*(v_2) \\ \vdots \\ x^*(v_N) \end{pmatrix} \in \boldsymbol{R}^N.$$

For a primitive extremal signature $\mathfrak{S}$ we define the linear subspace

$$V(\mathfrak{S}):=\{b \in \boldsymbol{R}^N \mid \forall_{x^* \in \mathfrak{S}} \langle b, V(x^*)\rangle=0\}.$$

The angle $\theta_v(\varepsilon)$ is defined like in the case $C_0(T)$. Then we have the following extension of Theorem 1:

Theorem 4. *Let v_0 be a best approximation to $x \in X\setminus V$ from V and let $\mathfrak{S}$ be a primitive extremal signature contained in $\mathfrak{S}_{x-v_0}$. Then there exists a real number $K_4>0$ such that*

$$\forall_{v \in V} \min_{x^* \in \mathfrak{S}_{x-v_0}} x^*(v) \leq -K_4 \|v\| \theta_\varepsilon^2(v).$$

The proof is similar to the proof of Theorem 1. To extend the strong uniqueness theorem, we introduce the set $\Gamma_{x-v_0} := U\{\mathfrak{E} \subset \mathfrak{E}_{x-v_0} \mid \mathfrak{E}$ primitive extremal$\}$, where v_0 is a best approximation to x from V. Then we have the following extension of Theorem 3:

Theorem 5. *An element v_0 in V is a strongly unique best approximation to x in X from V if and only if there exist linear functionals $x_1^*, x_2^*, \ldots, x_N^* \in \Gamma_{x-v_0}$ such that* $\det(x_\mu^*(v_\nu)) \neq 0$.

Also this proof is similar to the proof of Theorem 3. A simple consequence of Theorem 5 is the following result of Wulbert [5]:

Corollary 6. *In a smooth space X a best approximation from V is not strongly unique.*

Proof. In a smooth space $\mathfrak{E}_{x-v_0}$ is a singleton.

5. Generalization to Semi-Infinite Optimization Problems. In paper [3] the author generalized the criterion to semi-infinite optimization problems and applied it to parametric optimization problems. We give a brief description of the results in [3]. Let T be a compact Hausdorff-space and let $C(T)$ (resp. $C(T, \boldsymbol{R}^N)$) denote the set of all continuous mappings $b: T \to \boldsymbol{R}$ (resp. $T: \to \boldsymbol{R}^N$) normed by

$$\|b\|_\infty := \sup_{t \in T} |b(t)| \quad (\text{resp. } \|B\|_\infty := \sup_{t \in T} \|B(t)\|_2).$$

For each parameter $\sigma = (b, B)$ in $C(T)^{N+1}$, we consider the following linear minimization problem:

LM(σ). Minimize $\langle p, x\rangle := \Sigma_{\nu=1}^N p_\nu \cdot x_\nu$ subject to $\forall_{t \in T} \langle B(t), x\rangle \leq b(t)$, where $p_1, p_2, \ldots, p_N$ are given real numbers. For each parameter $\sigma = (b, B)$ we define the set of feasible points

$$Z_\sigma := \{x \in \boldsymbol{R}^N \mid \forall_{t \in T} \langle B(t), x\rangle \leq b(t)\},$$

the minimum value $E_\sigma := \inf\{p(x) \in \boldsymbol{R} \mid x \in Z_\sigma\}$ and the set of optimal solutions $P_\sigma := \{v \in Z_\sigma \mid p(v) = E_\sigma\}$.

Let t_0 be any point not in T. Then we define $B(t_0) := p$ and $b(t_0) := E_\sigma$. A closed subset $M \subset T$ is called extremal (with respect to $B: T \to \boldsymbol{R}^N$ and $p \in \boldsymbol{R}^N$) if

$$\forall_{v \in \boldsymbol{R}^N} \min_{t \in M \cup \{t_0\}} \langle B(t), v\rangle \leq 0.$$

An extremal subset $M \subset T$ is called primitive if for any closed subset $M' \subset M$ with $M' \neq M$ the subset M' is not extremal. Each primitive extremal subset is finite. For each parameter $\sigma = (b, B)$ and for each v in $\boldsymbol{R}^N$ we denote by $M_{b,v}$ the following closed set

$$M_{\sigma,v} := \{t \in T \mid \langle B(t), v\rangle - b(t) = \max_{s \in T \cup \{t_0\}} (\langle B(s), v\rangle - b(s))\}.$$

For each primitive extremal subset M and for each element v in $\boldsymbol{R}^N$ we denote by $\theta_v(M)$ the angle between v and the linear subspace

$$V(M) := \{x \in \boldsymbol{R}^N \mid \forall_{t \in M} \langle B(t), x\rangle = 0\}.$$

Then we have the following refinement of an optimality criterion of [2]:

Theorem 7. *Let the minimization problem* LM(σ) *be given and assume that the Slater-condition is fulfilled, i. e. there exists an element* $v_0 \in \boldsymbol{R}^N$ *such that* $\forall_{t \in T} \langle B(t), v_0 \rangle < b(t)$. *Let* v_0 *be in* P_σ *and let* M *be a primitive extremal subset contained in* M_{σ,v_0}. *Then, there exists a real number* $K_5>0$ *such that*

$$\forall_{v \in \boldsymbol{R}^N} \min_{t \in M_{b,v_0} \cup \{t_0\}} \langle B(t), v \rangle \leqq -K_5 \cdot \| v \|_2 \theta_v^2(M).$$

For the proof compare paper [3].

To generalize the strong uniqueness theorem and to obtain further results, we introduce the set $\Gamma_{\sigma,v} := \cup \{M \subset M_{\sigma,v} \mid M \text{ primitive extremal}\}$.

Theorem 8. *Let* $\sigma_0 = (b_0, B_0)$ *be a parameter such that the Slater-condition is fulfilled. Assume that there exist points* $t_1, t_2, \ldots, t_N$ in Γ_{σ_0,v_0} *such that* $B(t_1), B(t_2), \ldots, B(t_N)$ *are linearly independent.*

Then there exist real numbers $K_6, K_7, K_8 > 0$ *and an open set* $W \subset C(T)^{N+1}$ *with* $\sigma_0 \in W$ *such that* $\forall_{v \in Z_{\sigma_0}} \langle p,v \rangle \geqq \langle p,v_0 \rangle + K_6 \| v - v_0 \|_2$, *(strong uniqueness).*

For each parameter $\sigma = (b, B)$ *in* W *one has* $|E_\sigma - E_{\sigma_0}| \leq K_7(\| b - b_0 \|_\infty + \| B - B_0 \|_\infty)$. *For each parameter* $\sigma \in W$ *and each* $v \in P_\sigma$ *one has* $\| v - v_0 \|_2 \leqq K_5(\| b - b_0 \|_\infty + \| B - B_0 \|_\infty)$.

For the proof compare paper [3].

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