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INTERPOLATION OF OPERATORS AND APPROXIMATION OF FUNCTIONS

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Summary. The lecture contains the complete description of results briefly summarized in the article in *Doklady Akademii Nauk SSSR*, **256**, 1981, 14-17.

1. The main purpose of the lecture is the complete description of results briefly summarized in our article [1]. The article is the first part of the work; this part consists of the highly detailed investigation of some class of interpolation functors. Among the members of the class there are interpolation functors generating by the K-, J- and E-methods and so on. That is why this part is titled "Real Interpolation Functors". By our intention the second part of the work must have been an investigation of the families of functors generated by complex interpolation methods of Calderon and by real analogues of one constructed by Gagliardo, Peetre, Gustavsson and Ovchinnikov. The appearance of recent preprint of Janson [2] made this part of the work unnecessary. Time will show the advisability of writing the third part of the work.

2. There are some simple questions of approximation theory which motivated our search. We start with two results explaining the nature of these questions.

Let $\{A_n\}$ be increasing sequence of linear subspaces of $C[0,1]$ and let

$$\mathscr{E}(\varphi):=\{f:\sup_{n\geqq 0}[\varphi(n)\operatorname{dist}_C(f;A_n)]<\infty\}$$

be an approximation space.

Then, if $\omega:\boldsymbol{R}_+\to\boldsymbol{R}_+$ is a convex function, we set

$$H_k(\omega):=\{f\in C[0,1]:\ \sup_{t>0}\frac{\omega_k(f;t)}{\omega(t^k)}<\infty\}.$$

Theorem 1. *If $H_k(t^\alpha)=\mathscr{E}(t^{\alpha k})$ for some α, $0<\alpha\leqq 1$, then*

$$H_k(\omega)=\mathscr{E}(\omega^*)$$

with $\omega^*(t):=1/\omega(t^{-k})$ for any ω such that

$$\overline{\lim_{t\to 0}}\frac{\omega(2t)}{\omega(t)}\leqq 2^\alpha.$$

The theorem is intimately connected with the general form of the so-called reiteration theorem (see Theorem 7 below).

Theorem 2. *Let* $\{\psi_n : \boldsymbol{R}_+ \to \boldsymbol{R}_+\}$ *be a sequence of concave functions with* $\Sigma\psi_n < \infty$. *Consider function* $f \in C[0, 1]$ *for which*

$$\omega(f; t) \leq \Sigma\psi_n(t), \quad t \in \boldsymbol{R}_+.$$

Then there exists a sequence $\{f_n\} \subset C[0, 1]$ *such that* $f = \Sigma f_n$ *in* $C[0, 1]$ *and*

$$\omega(f_n; t) \leq \gamma\psi_n(t), \quad t \in \boldsymbol{R}_+,$$

for $n = 1, 2, \ldots$

Conjecture: $\inf \gamma = 2$.

We know only that $1 < \inf \gamma < 14$. Recently Podogova proved that $\inf \gamma > 3/2$.

Theorem 2 is a simple consequence of the urgent property of the K-functional (see Theorem 3 below).

3. Let us introduce some definitions (see [3] for more details). By a (Banach) couple $\mathbf{X} = (X_0, X_1)$ we mean an entity consisting of Banach spaces X_i both linearly and continuously embedded in some Hausdorff topological vector space. Therefore we can consider sum $\Sigma(\mathbf{X}) := X_0 + X_1$ and intersection $\Delta(\mathbf{X}) := X_0 \cap X_1$. Let us introduce K-functional on $\Sigma(\mathbf{X})$ by the formula

$$K(t; x; \mathbf{X}) := \inf\{\| x_0 \|_{X_0} + t \| x_1 \|_{X_1} : x = x_0 + x_1\}.$$

Suppose $\{\psi_n\}$ is a sequence as in Theorem 2.

Theorem 3. *For any* $x \in \Sigma(\mathbf{X})$ *for which*

$$K(t; x; \mathbf{X}) \leq \Sigma\psi_n(t), \quad t \in \boldsymbol{R}_+,$$

there is a sequence $\{x_n\} \subset \Sigma(\mathbf{X})$ *such that* $x = \Sigma x_n$ *in* $\Sigma(\mathbf{X})$ *and*

$$K(t; x_n; \mathbf{X}) \leq \gamma(\mathbf{X})\psi_n(t), \quad t \in \boldsymbol{R}_+,$$

for $n = 1, 2, \ldots$. *Moreover,*

$$1 < \sup \gamma(\mathbf{X}) < 14.$$

Conjecture: $\sup \gamma(\mathbf{X}) = 2$ and supremum is attained over $\mathbf{X} = (C, C^1)$.

For the formulation of other results we need some more definitions with interpolation theory (see, e. g., [3]). The important families of interpolation functors are generated by the K- and J-methods. The interpolation functor $\mathfrak{K}_\Phi$ is introduced by the formula

$$\mathfrak{K}_\Phi(\mathbf{X}) := \{x \in \Sigma(\mathbf{X}) : \| K(\cdot; x; \mathbf{X}) \|_\Phi < \infty\};$$

here Φ is any functional norm defined for all concave functions $\varphi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$. In particular case $\Phi = L_p^\theta$, where

$$\| \varphi \|_{L_p^\theta} := \{\int_{\boldsymbol{R}_+} | t^{-\theta}\varphi(t) |^p \frac{dt}{t}\}^{1/p}$$

we write $\mathbf{X}_{\theta p}$ instead of $\mathfrak{K}_\Phi(\mathbf{X})$.

The functor $\mathfrak{J}_\Psi$ is defined in dual way (see [3]).

Let introduce the couples $\mathbf{L}_\infty := (L_\infty^0, L_\infty^1)$ and $\mathbf{L}_1 := (L_1^0, L_1^1)$.

Definition 1 (Aronszajn, Gagliardo). Interpolation functor $\mathscr{F}$ is named maximal for the couple $\mathbf{A}$ if for any interpolation functor $\mathscr{Y}$ such that $\mathscr{Y}(\mathbf{A}) \subset \mathscr{F}(\mathbf{A})$ there are analogously embedding $\mathscr{Y}(\mathbf{X}) \subset \mathscr{F}(\mathbf{X})$ for all other couples $\mathbf{X}$.

Minimal interpolation functor for the couple $\mathbf{A}$ is defined in a dual way

It is surprising that the following fundamental result was established only in 1978.

Theorem 4. *$\mathfrak{K}_\Phi$ is maximal for $\mathbf{L}_\infty$ and $\mathfrak{J}_\Psi$ is minimal for $\mathbf{L}_1$.*

From this result we can without any loss of generality to assume that $\Phi(\Psi)$ in definition $\mathfrak{K}_\Phi(\mathfrak{J}_\Psi)$ is an interpolation space of the couple $\mathbf{L}_\infty(\mathbf{L}_1)$. We will assume this below.

Theorem 5 (First equivalence theorem).

(a) *If $\Psi \cap L_1^i \neq \Delta(\mathbf{L}_1)$, $\quad i=0,1$, then $\mathfrak{J}_\Psi = \mathfrak{K}_\Phi$ with $\Phi := \mathfrak{J}_\Psi(\mathbf{L}_\infty)$.*

(b) *If $\Psi \cap L_1^i = \Delta(\mathbf{L}_1)$ for some i but $\Psi \neq \Delta(\mathbf{L}_1)$, then $\mathfrak{J}_\Psi = \mathfrak{K}_\Phi \cap \mathscr{P}r_i$ with the same Φ. Here $\mathscr{P}r_i(\mathbf{X}) := X_i$.*

Remark 1. If $\Psi = \Delta(\mathbf{L}_1)$, then $\mathfrak{J}_\Psi = \Delta$.

The relations between $\mathfrak{K}_\Phi$ and $\mathfrak{J}_\Psi$ are more complicated. To formulate the result we mark closure $\Delta(\mathbf{X})$ in $\mathscr{F}(\mathbf{X})$ by $\mathscr{F}^0(\mathbf{X})$ and relative completion $\mathscr{F}(\mathbf{X})$ in $\Sigma(\mathbf{X})$ by $\mathscr{F}^c(\mathbf{X})$. Let us restrict the formulation by the main case $\Phi \subset \Sigma^0(\mathbf{L}_\infty)$. Another cases are consequences from this.

Theorem 6 (Second equivalence theorem).

(a) *If $\Phi \cap L_\infty^i \neq \Delta(\mathbf{L}_\infty)$, $i=0,1$, then $\mathfrak{K}_\Phi = \mathfrak{J}_\Psi$ with $\Psi := \mathfrak{K}_\Phi(\mathbf{L}_1)$.*

(b) *If $\Phi \cap L_\infty^i = \Delta(\mathbf{L}_\infty)$ for some i but $\Phi \neq \Delta(\mathbf{L}_\infty)$, then $\mathfrak{K}_\Phi = \mathfrak{J}_\Psi + (\mathscr{P}r_i \cap \Delta^c)$ with the same Ψ.*

Remark 2. If $\Phi = \Delta(\mathbf{L}_\infty)$, then $\mathfrak{K}_\Phi = \Delta^c$.

It is interesting to compare these results with the classical equivalence theorem of Peetre. Its modern variant states that

$$\mathfrak{K}_\Phi = \mathfrak{J}_\Phi \tag{1}$$

if and only if the operator

$$Sf := \int_{R_+} \min(1, t/x) f(x) \frac{dx}{x} \tag{2}$$

is bounded in Φ.

Unfortunately there are few functional norms Φ for which $\|S\|_\Phi < \infty$. That is why all generalizations which contain relation (1) do not give the essential extension of the theory.

Theorem 7 (Reiteration theorem).

$$\mathfrak{K}_\Phi(\mathfrak{K}_{\Phi_0}, \mathfrak{K}_{\Phi_1}) = \mathfrak{K}_{\overline{\Phi}}, \quad \mathfrak{J}_\Psi(\mathfrak{J}_{\Phi_0}, \mathfrak{J}_{\Psi_1}) = \mathfrak{J}_{\overline{\Psi}}.$$

Here $\overline{\Phi} := \mathfrak{K}_\Phi(\Phi_0, \Phi_1)$ and $\overline{\Psi} := \mathfrak{J}_\Psi(\Psi_0, \Psi_1)$.

Example 1. Let $\|f\|_{L_\infty^\omega} := \sup\limits_{t>0} \dfrac{|f(t)|}{\omega(t)}$ and $\omega : R_+ \to R_+$ is a concave function. Then

$$\mathfrak{K}_{L_\infty^\omega}(\mathfrak{K}_{L_\infty^{\omega_0}}, \mathfrak{K}_{L_\infty^{\omega_1}}) = \mathfrak{K}_{L_\infty^{\overline{\omega}}}$$

with $\overline{\omega} := \omega_0 \cdot \omega(\omega_1/\omega_0)$.

Theorem 8 (Density theorem). (a) *$\Delta(\mathbf{X})$ is dense in $\mathfrak{J}_\Psi(\mathbf{X})$ for any $\mathbf{X}$ if and only if $\Delta(\mathbf{L}_1)$ is dense in Ψ.*

(b) $\Delta(\mathbf{X})$ *is dense in* $\mathfrak{R}_\Phi(\mathbf{X})$ *if and only if* $\Delta(\mathbf{L}_\infty)$ *is dense in* Φ *and moreover the condition of Theorem* 6(a) *holds.*

Remark 3. If Ψ is not an interpolation space of $\mathbf{L}_1$, then the condition of (a) is supposed for density of $\Delta(\mathbf{X})$ in $\mathfrak{J}_\Psi(\mathbf{X})$.

Suppose that $\Delta(\mathbf{X})$ is dense in $\Sigma(\mathbf{X})$ and therefore the conjugated couple $\mathbf{X}^* := (X_0^*, X_1^*)$ exists.

Let Φ' be Banach space dual to Φ under bilinear form

$$\langle f, g\rangle := \int_{R_+} f(t)\, g(1/t)\, \frac{dt}{t}.$$

Theorem 9 (Duality theorem).

(a) $\mathfrak{J}_\Psi^0(\mathbf{X})^* = \mathfrak{R}_{\Psi'}(\mathbf{X}^*)$.

(b) *If the condition of Theorem* 6 (a) *holds for* Φ, *then*

$$\mathfrak{R}_\Phi^0(\mathbf{X})^* = \mathfrak{J}_{\Phi'}(\mathbf{X}^*).$$

Remark 4. The assertion (a) is valid without the condition that is interpolation space of

4. Some applications.

(a) We start with

Theorem 10. *If an interpolation functor* $\mathscr{F}$ *coincides with* $\mathfrak{R}_\Phi$ *in the couples* $\mathbf{L}_\infty$ *and* $\mathbf{L}_1$, *then*

$$\mathscr{F} + \Delta^c = \mathfrak{R}_\Phi.$$

For many situations $\mathscr{F} + \Delta^c = \mathscr{F}$ and as consequence we have the simple formula

$$\mathscr{F}(\mathbf{X}) = \mathfrak{R}_{\mathscr{F}(\mathbf{L}_\infty)}(\mathbf{X}). \tag{3}$$

(3) in particular holds for relatively complete couples (i. e. in case $\Delta^c(\mathbf{X}) = \Delta(\mathbf{X})$) or for arbitrary couples if

$$\mathscr{F}(\mathbf{L}_\infty) \cap L_\infty^i \neq \Delta(\mathbf{L}_\infty), \quad i = 0, 1. \tag{4}$$

Example 2. Let $\mathfrak{J}_{\Phi_0, \Phi_1}$ is an interpolation functor generated by the so-called method of means (see, e. g. [4]). Then under condition (4) with $\mathscr{F} = \mathfrak{J}_{\Phi_0, \Phi_1}$ we see that $\mathfrak{J}_{\Phi_0, \Phi_1}$ coincides with $\mathfrak{R}_\Phi$.

Example 3. Let

$$E(t; x; \mathbf{X}) := \inf\{\| x - y \|_{X_0} : \| y \|_{X_1} \leq t\}$$

be E-functional of $x \in \Sigma(\mathbf{X})$. We define interpolation functor $\mathscr{E}_\Phi$ by the norm

$$\| x \|_{\mathscr{E}_\Phi(\mathbf{X})} := \inf\{\lambda > 0 : \| E(\cdot; \lambda^{-1} x; \mathbf{X}) \|_\Phi \leq 1\}.$$

The relation (3) holds for $\mathscr{E}_\Phi$ with equality of norms

$$\mathscr{E}_\Phi \overset{\text{iso}}{=\!=} \mathfrak{R}_{\widehat{\Phi}}, \tag{5}$$

where $\widehat{\Phi}$ makes up from Φ by Legendre-Young transform

$$f^\frown(t) := \sup\{f(s) - ts : s > 0\}.$$

Namely,

$$\| f \|_{\widehat{\Phi}} := \inf\{\lambda > 0 : \| (\lambda^{-1} f)^\frown \|_\Phi \leq 1\}.$$

The relation (5) contains the classical Peetre-Sparr result and some new results which cannot be approached by the method of these authors.

(b) Gagliardo problem on quasilinear interpolation.

Definition 2 (Gagliardo, Peetre). (Nonlinear) map $T:\Sigma(\mathbf{X})\to\Sigma(\mathbf{Y})$ is named quasilinear operator from **X** into **Y** (briefly: $T\in Q(\mathbf{X},\mathbf{Y})$) if such a constant $\gamma>0$ exists that for any $x_i\in X_i$ and any $\varepsilon>0$ there are $y_i\in Y_i$, for which

$$T(x_0+x_1)=y_0+y_1, \quad \|y_i\|_{Y_i}\leq\gamma\|x_i\|_{X_i}+\varepsilon.$$

Gagliardo problem consists of description of all Q-invariant spaces of any couple **X***). The complete decision of Gagliardo problem contains the following

Theorem 10. *Triplet X,* **X** *is Q-invariant relatively of triplet Y,* **Y** *if and only if there exists a function norm* Φ *such that*

$$X\subset\mathfrak{R}_\Phi(\mathbf{X}),\ Y\supset\mathfrak{R}_\Phi(\mathbf{Y}).$$

Corollary 1. *The family* $\{\mathfrak{R}_\Phi(\mathbf{X})\}$ *coincides with the set all Q-invariant spaces of couple* **X**.

(c) The problem of commutativity interpolation functors.

Suppose that one of the following conditions holds

(i) Couple **X** is relatively complete;

(ii) $\Phi_i\subset\Sigma^0(\mathbf{L}_\infty)$, $i=0,1$, and $\Phi_i\cap\mathbf{L}_\infty^k\neq\Delta(\mathbf{L}_\infty)$, $i,k=0,1$.

In this case the following result is valid.

Theorem 11. *The equality*

$$\mathscr{F}(\mathfrak{R}_{\Phi_0}(\mathbf{X}),\mathfrak{R}_{\Phi_1}(\mathbf{X}))=\mathfrak{R}_{\mathscr{F}(\Phi_0,\Phi_1)}(\mathbf{X})$$

takes place if and only if it is true for $\mathbf{X}=\mathbf{L}_1$.

Corollary 2. *If the operator* $\int$ *(see (2)) is bounded in couple* $\Phi=(\Phi_0,\Phi_1)$, *then*

$$\mathscr{F}(\mathfrak{R}_{\Phi_0},\mathfrak{R}_{\Phi_1})=\mathfrak{R}_{\mathscr{F}(\Phi_0,\Phi_1)}.$$

As consequence of the corollary we can receive the well-known results of Lions-Grisvard-Karadjov [5] and Cwickel-Dmitriev-Ovchinnikov [6].

5. Some results of the preceding sections are extended with some restrictions on the situation of couples of quasinormal Abelian groups (see [3]). Thus Theorem 3 is valid in this case only for finite sequences $\{\psi_n\}$. Also the first assertion of Theorem 7 holds for couples of quasinormed Abelian groups, etc.

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*) Triplet X, **X** is Q-invariant relatively of triplet Y, **Y** if $T(X)\subset Y$ for all $T\in Q(\mathbf{X},\mathbf{Y})$.

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