КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'81
София, 1983, с. 258—273

CONSTRUCTIVE FUNCTION THEORY'81
Sofia, 1983, p. 258—273

THE SHANNON SAMPLING THEOREM AND SOME OF ITS GENERALIZATIONS. AN OVERVIEW *

P. L. Butzer

Summary. The aim of this paper is to present an overview of results concerning the Whittaker-Kotelnikov-Shannon sampling theorem and its various extensions obtained at Aachen since 1977. This theorem, basic in communication theory, is often called the cardinal interpolation series theorem in mathematical circles. Emphasis is placed on error analysis, including the aliasing, round-off (or quantization), and time jitter errors. Some new error estimates are given, others are improved, many of the proofs are reduced to a common structure. Both deterministic and probabilistic methods are employed.

1. Introduction. The well-known Shannon [39] sampling theorem which plays a basic role in communication, control theory and data processing, states that every real-valued signal function $f(t)$ that is bandlimited to $[-\pi W, \pi W]$, $W>0$, can be completely reconstructed from its values (samples $f(k/W)$ taken at the nodes k/W equally spaced apart on the real axis $\boldsymbol{R}$, in terms of

$$(1.1)\quad f(t)=\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}=\sin(\pi Wt)\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\frac{(-1)^k}{\pi(Wt-k)}\quad (t\in\boldsymbol{R})$$

($\Sigma_{k=-\infty}^{\infty}$ being understood as $\lim_{n,m\to\infty}\Sigma_{k=-n}^{m}$, usually with $m=n$). The latter form of this theorem recalls to mind that it had been considered much earlier by the mathematicians de La Vallée Poussin [55], E. T. Whittaker [57], Ferrar [25] and J. M. Whittaker [58, 59]; the sampling series was called the cardinal interpolation series by them.

Returning to the engineering literature, Kotelnikov [28] had considered the theorem earlier than, and Someya [41] parallel to C. Shannon. In any case, between 1950 and 1975 at least 250 articles dealing with various aspects of the sampling theorem, written by about 170 different authors, appeared in engineering journals (see the survey paper by Jerri [27] and the historical report by Lüke [30]).

But from the point of view of pure mathematics there is only a small number of papers on the subject, e. g. by Campbell [20], J. L. Brown

* An expanded version of a lecture held at the International Conference on Constructive Function Theory, Golden Sands (Varna), Bulgaria, on June 1, 1981.

[7], McNàmee, Stenger and Whitney [32], Boas [5], Pollard and Shisha [34], Boas and Pollard [6], Sofman [40], Buslaev and Vituškin [10], Vituškin [56], Stenger [51], who followed up the work of the four pioneering mathematicians mentioned. Although the material should be a standard topic in books on Fourier analysis, just the books by Boas [4], Schönhage [38], Dym and McKean [22], and Triebel [54] seem to cover the sampling theorem.

The aim of my talk is to try to draw the sampling theorem to the attention of a wider group of mathematicians, for it indeed belongs to the interdisciplinary domain of Fourier analysis, interpolation, approximation and communication engineering. For this purpose I will mainly report on a few of the developments in connection with this theorem at Aachen since 1977. These were carried out chiefly by Dr. W. Splettstösser, but also by Drs. R. L. Stens, G. Wilmes and the present author. At the same time most of the results are presented in a sharpened form, from a fresh point of view, or with new proofs. As a matter of fact, all but one of the proofs have a common structure. Theorems 7 and 9b) seem to be new.

Let me first look at the hypotheses of the theorem. It is generally assumed that f belongs to $C(\boldsymbol{R})$ (=class of functions which are uniformly continuous and bounded on the real axis $\boldsymbol{R}$) and to $L(\boldsymbol{R})$ (=class of functions which are absolutely integrable over $\boldsymbol{R}$ in Lebesgue's sense). That $f(t)$ is bandlimited to $[-\pi W, \pi W]$, some $W>0$, means that the Fourier transform of $f\in L(\boldsymbol{R})$, namely $f^{\wedge}(v):=(1/\sqrt{2\pi})\int_{\boldsymbol{R}} f(u)e^{-ivu}du$, $v\in\boldsymbol{R}$, vanishes for all $|v|>\pi W$. In that case the Fourier inversion theorem gives

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\pi W}^{\pi W} f^{\wedge}(v)e^{itv}\,dv \quad (t\in\boldsymbol{R}). \tag{1.2}$$

The sampling theorem may now be stated more precisely.

Theorem 1. *If $f\in C(\boldsymbol{R})\cap L(\boldsymbol{R})$, and f is bandlimited to $[-\pi W, \pi W]$, then the representation* (1.1) *holds for each $t\in\boldsymbol{R}$, the series being absolutely and uniformly convergent on $\boldsymbol{R}$.*

If $f\in C(\boldsymbol{R})\cap L(\boldsymbol{R})$, then $f\in L^2(\boldsymbol{R})$ (=class of functions quadratically integrable over $\boldsymbol{R}$). In this regard, the Paley-Wiener theorem states that any $f\in L^2(\boldsymbol{R})$ has the representation (1.2) iff f is the restriction to $\boldsymbol{R}$ of an entire function F of exponential type πW, i. e.,

$$|F(z)|\leq e^{\pi W|y|}\,\|F\|_C \quad (z=x+iy\in\mathscr{C}),$$

where $\|F(\cdot)\|_C:=\|F\|_C:=\sup_{t\in\boldsymbol{R}}|F(t)|$.

Concerning the series (1.1) itself, it is of interest that it interpolates f at the nodes $t=k/W$ just because

$$\operatorname{si}\{\pi(Wt-k)\}:=\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}=\begin{cases}1, & t=k/W,\\ 0, & t=l/W,\ k\neq l\in Z.\end{cases}$$

This leads to a "formal" proof of (1.1): if it would be known that the sum in the first series of (1.1), regarded as a convolution sum of f and the si-function, be commutative, then it would be equal to

$$\sum_{k=-\infty}^{\infty} f\Big(t-\frac{k}{W}\Big)\frac{\sin\pi k}{\pi k}=f(t) \quad (t\in\boldsymbol{R}).$$

2. Sampling Expansions of Non-Bandlimited Functions. Practice demands that one also tries to consider representations of type (1.1) for duration-limited functions, i. e., for functions $f\in C(\mathbf{R})$ such that $f(t)=0$ for all $|t|>T$, some $T>0$. Since such functions cannot be simultaneously bandlimited (unless they vanish everywhere), in view of the Paley-Wiener theorem, one needs to extend the sampling theorem to not necessarily bandlimited functions. Note that those functions $f\in L(\mathbf{R})$ that are either bandlimited or duration-limited form a dense subspace of $L(\mathbf{R})$.

This leads to the following theorem, considered by Brown [7], Boas [5], and Butzer and Splettstösser [15].

Theorem 2. *If $f\in C(\mathbf{R})\cap L(\mathbf{R})$ and $f^{\wedge}\in L(\mathbf{R})$, then, uniformly in $t\in\mathbf{R}$,*

$$f(t)=\lim_{W\to\infty}\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}. \tag{2.1}$$

It states that $f(t)$ can be reconstructed from its sampling sum provided one takes its limit as $W\to\infty$. Here the distance between the nodes k/W is not fixed as in Theorem 1 but decreases for $W\to\infty$, so that the number of nodes increases. Thus the series in (2.1) approximates and simultaneously interpolates $f(t)$ at $t=k/W$ for each fixed W.

Let me prove Theorems 1 and 2 by means of a slight modification of the proof of Boas [5], using results which directly precede the basic Poisson summation formula (cf. [14, pp. 123 f, 201 ff.]). Let $f, f^{\wedge}\in L(\mathbf{R})$. These state that if

$$F^*(v):=\sqrt{2\pi}W\sum_{k=-\infty}^{\infty} f^{\wedge}(2k\pi W-v), \tag{2.2}$$

then this series is dominatedly convergent on every compact interval, and so $F^*\in L_{2\pi W}$, $[F^*]^{\wedge}_{C}(k)=[f^{\wedge}]^{\vee}(-k/W)=f(k/W)$ $(k\in\mathbf{R})$, where $[g]^{\wedge}_{C}(k):=(1/2\pi W)\int_{-\pi W}^{\pi W} g(u)\exp(-iku/W)du$, $k\in\mathbf{R}$, denote the Fourier coefficients of $g\in L_{2\pi W}$ (=class of functions which are $2\pi W$-periodic and Lebesgue integrable over $(-\pi W, \pi W)$). So one has the Fourier expansion

$$F^*(v)\sim\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)e^{ikv/W}\quad (v\in\mathbf{R}). \tag{2.3}$$

Since a Fourier series can be multiplied by e^{-ivt}, a function of bounded variation, and integrated term by term (see [61, p. 160]), and

$$\frac{1}{2\pi W}\int_{-\pi W}^{\pi W} e^{i(k/W-t)v}dv=\operatorname{si}\{\pi(Wt-k)\}\quad (k\in\mathbf{R}),$$

it follows that (2.3) yields for each $t\in\mathbf{R}$,

$$\frac{1}{2\pi W}\int_{-\pi W}^{\pi W} F^*(v)e^{-ivt}dv=\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\operatorname{si}\{\pi(Wt-k)\}=:S(t).$$

Replacing the function F^* in the integral on the left by the series (2.2), interchanging integral and sum (possible by the dominated convergence), and substituting $2k\pi W-v$ by v, then

$$S(t)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} e^{-i2k\pi Wt}\int_{(2k-1)W\pi}^{(2k+1)W\pi} f^{\wedge}(v)e^{ivt}dv.$$

On the other hand, one has

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^{\wedge}(v)e^{ivt}dv=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\pi W}^{(2k+1)\pi W} f^{\wedge}(v)e^{ivt}dv.$$

This gives for each $t\in \mathbf{R}$

$$(2.4)\quad |(R_W f)(t)|:=|f(t)-S(t)|=\frac{1}{\sqrt{2\pi}}\left|\sum_{k=-\infty}^{\infty}(1-e^{-i2k\pi Wt})\int_{(2k-1)\pi W}^{(2k+1)\pi W} f^{\wedge}(v)e^{ivt}dv\right|$$

$$\leq\frac{2}{\sqrt{2\pi}}\sum_{k\neq 0}\int_{(2k-1)\pi W}^{(2k+1)\pi W}|f^{\wedge}(v)|\,dv=\frac{2}{\sqrt{2\pi}}\int_{|v|>\pi W}|f^{\wedge}(v)|\,dv.$$

Now, if $f^{\wedge}$ vanishes outside $[-\pi W, \pi W]$, then $(R_W f)(t)=0$ for all $t\in \mathbf{R}$ proving Theorem 1. In general, if $f^{\wedge}\in L(\mathbf{R})$, then $\lim_{W\to\infty}(R_W f)(t)=0$ uniformly in $t\in \mathbf{R}$, establishing Theorem 2.

The above estimate is, as Brown [7] shows, the best result of its kind. For a quite different proof the reader may consult Butzer-Splettstösser [15, 16].

It is possible to weaken the hypotheses of Theorem 2 slightly.

Theorem 2*. *If* $f(t)=(1/\sqrt{2\pi})\int_{\mathbf{R}} g(v)e^{itv}dv$, $t\in \mathbf{R}$, *with* $g\in L(\mathbf{R})$, *then* (2.1) *holds uniformly in* $t\in \mathbf{R}$.

Indeed, if $f, f^{\wedge}\in L(\mathbf{R})$, then $g(v)=f^{\wedge}(v)$, $v\in \mathbf{R}$.

3. Sampling Theorem for Duration-Limited Functions; Comparison with Classical Results. If a function f is to be determined by its sampled values $f(k/W)$, $k\in Z$ in case it is duration-limited, $2N+1$ such values must be evaluated; here $N=N(T, W):=[TW]$ is the largest integer equal to or less than TW. Indeed,

Theorem 3. *Let* $f\in C(\mathbf{R})$ *be such that* $f(t)=0$ *for all* $|t|>T$ *and* $f^{\wedge}\in L(\mathbf{R})$. *Then, uniformly in* $t\in \mathbf{R}$,

$$(3.1)\qquad f(t)=\lim_{W\to\infty}\sum_{k=-N}^{N} f\left(\frac{k}{W}\right)\frac{\sin \pi(Wt-k)}{\pi(Wt-k)}.$$

The result follows from Theorem 2 by noting that $f(\pm k/W)$ vanishes for $|k|>N$, but does not for $|k|\leq N$.

Let us reformulate Theorem 3 so that it is comparable with well-known results on interpolation. Setting $C_{2\pi}(\mathbf{R}):=\{f\in C(\mathbf{R});\ f(t)=0$ for $t\notin[0, 2\pi]\}$, an application of Theorem 2 with $W=(2n+1)/2\pi$ gives, together with

$$(3.2)\quad (T_n f)(t):=\frac{2}{2n+1}\sum_{k=0}^{2n} f\left(\frac{2\pi}{2n+1}k\right)\left\{\sin\frac{2n+1}{2}\left(t-\frac{2\pi}{2n+1}k\right)\right\}\Big/\left(t-\frac{2\pi}{2n+1}k\right)\quad (n\in \mathbf{N}).$$

Corollary 1. *Let* $f\in C_{2\pi}(\mathbf{R})$ *such that* $f^{\wedge}\in L(\mathbf{R})$. *Then, uniformly in* $t\in \mathbf{R}$,

$$(3.3)\qquad f(t)=\lim_{n\to\infty}(T_n f)(t).$$

The operators $T_n: C_{2\pi}(\boldsymbol{R}) \to C(\boldsymbol{R})$, defined by (3.2), are bounded for each fixed $n \in \boldsymbol{N}$ but not uniformly so since the operator norms are divergent. Indeed (cf. [52]),

$$\| T_n \|_{[C_{2\pi}(\boldsymbol{R}), C(\boldsymbol{R})]} = \sup_{t \in \boldsymbol{R}} \frac{2}{2n+1} \sum_{k=1}^{2n} \{\sin \frac{2n+1}{2}(t - \frac{2\pi}{2n+1}k)\}/(t - \frac{2\pi}{2n+1}k) \simeq \log n.$$

The fact that $(T_n f)(t)$ converges uniformly to $f(t)$ for $n \to \infty$ is no contradiction to the Banach-Steinhaus theorem since in addition to $f \in C_{2\pi}(\boldsymbol{R})$ it is assumed that $f^{\wedge} \in L(\boldsymbol{R})$. Note that T_n is not a polynomial operator, nor does it define a periodic function.

In comparison, let me recall the trigonometric Lagrange interpolating polynomial of $f \in C_{2\pi}$ (=class of functions which are continuous and 2π-periodic on $\boldsymbol{R}$) at the equidistant nodes $t_k := (2\pi/(2n+1))k$, $0 \leq k \leq 2n$, $n \in \boldsymbol{P}$ (cf. Zygmund [61, II, p. 4 ff]). It can be written as

$$(L_n f)(t) = \frac{2}{2n+1} \sum_{k=0}^{2n} f(\frac{2\pi}{2n+1}k)\{\sin \frac{2n+1}{2}(t - \frac{2\pi}{2n+1}k)\}/(2 \sin \frac{1}{2}(t - \frac{2\pi}{2n+1}k)) \quad (t \in \boldsymbol{R}).$$

This polynomial interpolates f at $t = t_k$: $(L_n f)(t_k) = f(t_k)$. Moreover, $(L_n p_n)(t) = p_n(t)$ for any trigonometric polynomial $p_n \in \mathscr{P}_n$ (=class of all p_n of degree $\leq n$). For each fixed $n \in \boldsymbol{N}$ L_n is a bounded, linear operator mapping $C_{2\pi}$ onto $\mathscr{P}_n$ which is idempotent. So on account of the Harsiladze-Losinskiĭ theorem (cf. [21]) it cannot be expected that $(L_n f)(t)$ converges uniformly to $f(t)$ for every $f \in C_{2\pi}$ unless f satisfies in addition some smoothness condition (such as (4.9); cf. [38, § 5.4]). For L_n one has $\| L_n \|_{[C_{2\pi}, \mathscr{P}_n]} \cong \log n$.

4. Error Estimates for Non-Bandlimited Functions. Our next question is the rate of convergence in Theorem 2, in other words, how good is the approximation of $f(t)$ by the sum in (2.1), namely $(R_W f)(t)$ of (2.3) — often called the aliasing error — in dependence upon smoothness conditions on f. These are primarily given in terms of Lipschitz classes. Such a class of order α, $0 < \alpha \leq 1$, is defined by

$$\text{(4.1)} \qquad \mathrm{Lip}_L(\alpha; C) := \{ f \in C(\boldsymbol{R});\ \sup_{|h| \leq \delta} \| f(\cdot + h) - f(\cdot) \|_C \leq L\delta^{\alpha},\ \alpha > 0 \}.$$

Secondly, f is assumed to have a given rate of decay at infinity, $f(t) = O(|t|^{-\gamma})$ for $|t| \geq t_0$, some $0 < \gamma \leq 1$. Since this assumption is trivial for $t \leq t_0$ if $f \in C(\boldsymbol{R})$, it is equivalent to

$$\text{(4.2)} \qquad |f(t)| \leq M_f |t|^{-\gamma} \quad (t \neq 0)$$

for some $0 < \gamma \leq 1$. Note that if $f \in C_{2\pi}(\boldsymbol{R})$, then $f \in L(\boldsymbol{R})$ and $|f(t)| \leq \| f \|_C 2\pi / |t|$ for $t \neq 0$, i. e. (4.2) is satisfied with $\gamma = 1$.

For our theorem in this regard, due basically to Splettstösser [43] and Stens [52], two lemmas, contained implicitly in [50], will be needed.

Lemma 1. *One has for $q > 1$, $1/p + 1/q = 1$, $W > 0$,*

$$\Sigma_{k=-\infty}^{\infty} |\mathrm{si}\{\pi(Wt - k)\}|^q \leq 1 + (\frac{2}{\pi})^q \frac{q}{q-1} < p.$$

Lemma 2. *Assume that condition* (4.2) *is satisfied for some* $\gamma \in (0,1]$. *For each $p \geq 2/\gamma$, $W > 0$, $V > 0$,*

$$(\sum_{|k| > V} |f(k/W)|^p)^{1/p} \leq M_f (\sum_{|k| > V} |\frac{k}{W}|^{-\gamma p})^{1/p} \leq 2^{1/p} M_f W^{\gamma} V^{(1-p\gamma)/p}.$$

Theorem 4. *Let $f\in C(\mathbf{R})\cap L(\mathbf{R})$ satisfy* (4.2) *for some* $0<\gamma\leqq 1$. *If $f^{(r)}\in \mathrm{Lip}_L(\alpha, C)$ for $0<\alpha\leqq 1$, $r\in \mathbf{P}$, then*

$$(4.3)\qquad \|(R_W f)(\cdot)\|_C := \|f(\cdot)-\sum_{k=-\infty}^{\infty} f(\tfrac{k}{W})\,\mathrm{si}\{\pi(W\cdot -k)\}\|_C$$

$$\leqq M_1(f, r, \alpha, \gamma) W^{-r-\alpha}\log W$$

provided $W\geqq \exp\{2/(r+\alpha+\gamma)\}$, *with constant* M_1 *given in* (4.7).

Concerning the proof, first consider the de La Vallée Poussin means, defined for $f\in C(\mathbf{R})$, $t\in \mathbf{R}$, $\rho>0$ by

$$(4.4)\qquad (VP_\rho f)(t):=\frac{3\rho}{2\pi}\int_{\mathbf{R}} f(t-u)\,\mathrm{si}\{\tfrac{3}{2}\rho u\}\,\mathrm{si}\{\tfrac{1}{2}\rho u\}\,du.$$

They belong to $C(\mathbf{R})\cap L(\mathbf{R})$, they are known (cf. [50]) to satisfy the property (4.2) with $M_f':=3(M_f+\|f\|_C)$ for some $0<\gamma\leqq 1$ and $\rho\geqq 1$ provided f does. If $f^{(r)}\in \mathrm{Lip}_L(\alpha;\ C)$, then $\|f(\cdot)-(VP_\rho f)(\cdot)\|_C\leqq 7L\rho^{-r-\alpha}$ for $\rho\geqq 1$. Moreover, they are bandlimited to $[-2\rho, 2\rho]$. By Theorem 1 this implies that

$$(4.5)\qquad (VP_\rho f)(t)=\sum_{k=-\infty}^{\infty}(VP_\rho f)(\tfrac{k}{W})\,\mathrm{si}\{\pi(wt-k)\}$$

for $\rho=\pi W/2$ and each $t\in \mathbf{R}$. Hence

$$(R_W f)(t)=f(t)-(VP_\rho f)(t)+\sum_{k=-\infty}^{\infty}\{(VP_\rho f)(\tfrac{k}{W})-f(\tfrac{k}{W})\}\,\mathrm{si}\{\pi(Wt-k)\}$$

$$=: I_1(t)+I_2(t).$$

Since $\|I_1(\cdot)\|_C\leqq c_1 W^{-r-\alpha}$ for $W\geqq 1$, where $c_1:=7L(2/\pi)^{r+\alpha}$, to complete the proof it suffices to show that $\|I_2(\cdot)\|_C=O((\log W)/W^{r+\alpha})$. Indeed, Hölder's inequality yields

$$(4.6)\qquad |I_2(t)|\leqq \Big(\sum_{k=-\infty}^{\infty}|\mathrm{si}\{\pi(Wt-k)\}|^q\Big)^{1/q}\Big(\sum_{k=-\infty}^{\infty}|(VP_\rho f)(\tfrac{k}{W})-f(\tfrac{k}{W})|^p\Big)^{1/p}.$$

Split up the second sum into those $k\in \mathbf{Z}$ with $|k|\leqq V$, denoting it by I_2^1, and into those with $|k|>V$, denoting it by I_2^2, where V is to be chosen suitably. Then $I_2^1\leqq (2V+1)^{1/p}\|I_1(\cdot)\|_C$. Choosing $V:=[W^{1+(r+\alpha)/\gamma}+1]$, then $V\geqq 2$ for $W\geqq 1$, and

$$(2V+1)^{1/p}\leqq (\tfrac{5}{2})^{1/p}2^{1/p}\exp\{\tfrac{1}{p}(\tfrac{r+\alpha+\gamma}{\gamma})\log W\}=5^{1/p}e$$

provided $p:=((r+\alpha+\gamma)/\gamma)\log W$. Then $I_2^1\leqq 5^{1/p}e\,c_1 W^{-r-\alpha}$. Furthermore, since f satisfies (4.2), Lemma 2 gives

$$I_2^2\leqq c_2\Big(\sum_{|k|>V}|\tfrac{k}{W}|^{-\gamma p}\Big)^{1/p}\leqq c_2 2^{1/p}W^{\gamma}V^{1/p}V^{-\gamma}\leqq c_2 2^{1/p}2^{1/p}e W^{-r-\alpha}$$

with $c_2:=4M_f+3\|f\|_C$, provided $p\gamma\geqq 2$. The latter condition is equivalent to $\log W\geqq 2/(r+\alpha+\gamma)$ or $W\geqq \exp(2/(r+\alpha+\gamma))$.

Since the first sum in (4.6) is bounded by $p^{1/q}<p$ by Lemma 1, a combination of all estimates delivers

$$(4.7)\qquad \|(R_W f)\|_C \le \frac{c_1}{W^{r+\alpha}} + \left(\frac{r+\alpha+\gamma}{\gamma}\right)\{5^{\gamma/2}ec_1 + 2^{\gamma}c_2 e\}\frac{\log W}{W^{r+\alpha}}$$

$$\le \frac{(r+\alpha+\gamma)}{2\gamma}\{\gamma c_1 + 2\,.\,5^{\gamma/2}ec_1 + 2^{\gamma+1}ec_2\}\frac{\log W}{W^{r+\alpha}}$$

provided $W \ge \exp(2/(r+\alpha+\gamma))$. This proves the theorem.

Note that the concrete estimate in (4.7) of M_1 delivered by the above proof is the sharpest known so far. The hypothesis $f \in L(\boldsymbol{R})$ was used in the proof only to establish the validity of (4.5). But (4.5) is known to hold provided $f \in C(\boldsymbol{R})$ satisfies (4.2) (see [50]).

Theorem 3 or Corollary 1 may also be supplied with rates. Indeed, Theorem 4 yields

Corollary 2. *If* $f \in C_{2\pi}(\boldsymbol{R})$ *and* $f^{(r)} \in \mathrm{Lip}(\alpha;\, C)$ *for* $0 < \alpha \le 1$, $r \in \boldsymbol{P}$, then

$$\|f(\cdot) - \frac{2}{2n+1}\sum_{k=0}^{2n} f\left(\frac{2\pi}{2n+1}k\right)\{\sin\frac{2n+1}{2}\left(\cdot - \frac{2\pi}{2n+1}k\right)\}/\left(\cdot - \frac{2\pi}{2n+1}k\right)\|_C$$

$$= O\left(\frac{\log n}{n^{r+\alpha}}\right), \qquad (n \to \infty).$$

A weaker form of this corollary, namely under the additional hypothesis that the transform $f^{\wedge}$ belongs to $L(\boldsymbol{R})$, and the weaker order $o(n^{-r-\alpha-1})$, was established earlier by Butzer-Splettstösser [15, 16]. Moreover, Corollary 2 also holds for $r=0$, thus for functions which need not be differentiable at all. The estimate in Corollary 2 may also be stated in terms of the r-th modulus of continuity; see Butzer [11], where the proof follows as an application of the Banach-Steinhaus theorem with rates.

As an example, consider the function (see [52])

$$f(t) := \begin{cases} \sin t, & \text{for } t \in [0, 2\pi], \\ 0, & \text{for } t \text{ otherwise.} \end{cases}$$

Noting that $f \in \mathrm{Lip}(1;\, C)$, an application of Corollary 2 yields for $n \to \infty$

$$\|f(\cdot) - \frac{2}{2n+1}\sum_{k=0}^{2n}\sin\left(\frac{2\pi}{2n+1}k\right)\cdot[\sin\frac{2n+1}{2}\left(\cdot - \frac{2\pi}{2n+1}k\right)]/\left(\cdot - \frac{2\pi}{2n+1}k\right)\|_C = O\left(\frac{\log n}{n}\right).$$

If one would instead take the function

$$g(t):\ = \int_0^t f(u)du = \begin{cases} 1-\cos t, & t \in [0, 2\pi], \\ \quad 0, & t \text{ otherwise,} \end{cases}$$

then the error would be of order $O(\log n/n^2)$.

It is of interest to compare the assertions of Corollaries 1 and 2 with the corresponding ones in the case of the partial sums of the Fourier series of $f \in C_{2\pi}$, defined by

$$(4.8)\qquad (S_n f)(t) := \frac{1}{2\pi}\int_0^{2\pi} f(u)\left((\sin\frac{2n+1}{2}(t-u))/(\sin\frac{1}{2}(t-u))\right)du \quad (t \in \boldsymbol{R}).$$

In this regard, if $f \in C_{2\pi}$ satisfies the Dini-Lipschitz condition, i. e.

$$(4.9)\quad \omega(f;\, C_{2\pi}) := \sup\{\|f(\cdot+h) - f(\cdot)\|_{C_{2\pi}};\ |h| \le \delta\} = o(|1/\log\delta|)\ (\delta \to 0+),$$

then it is known that (cf. [14, p. 105]) $\lim_{n\to\infty}(S_n f)(t)=f(t)$ uniformly in $t\in \boldsymbol{R}$. Whereas Corollary 1 is a rough counterpart to this result, an "exact" one also holds. Indeed Stens [52] improved Corollary 1 (actually the proof of Corollary 2) to the effect that $f\in C_{2\pi}(\boldsymbol{R})$ together with $\sup\{\|f(\cdot+h)-f(\cdot)\|_C;\ |h|\leq\delta\}=o(|1/\log\delta|)$ $(\delta\to 0+)$ implies validity of (3.3).

The exact counterpart of Corollary 2 for Fourier series, actually its model, states that for any $f\in C_{2\pi}$ satisfying $\omega(f^{(r)};C_{2\pi})=O(\delta^\alpha)$, $\delta\to 0+$, for $0<\alpha\leq 1$, $r\in \boldsymbol{P}$, one has

$$\|f(\cdot)-(S_n f)(\cdot)\|_{C_{2\pi}}=o(n^{-r-\alpha}\log n)\ (n\to\infty). \tag{4.10}$$

One could also compare Theorem 4 with the corresponding one for Fourier integrals. In this respect, Splettstösser [47] has just shown that

$$\|f(\cdot)-W\int_{\boldsymbol{R}} f(u)\frac{\sin\pi W(\cdot-u)}{\pi W(\cdot-u)}du\|_C=O\left(\frac{\log W}{W^{r+\alpha}}\right), \tag{4.11}$$

for $W\to\infty$ provided $f\in C(\boldsymbol{R})$ satisfies (4.2) for $0<\gamma\leq 1$ and $f^{(r)}\in \mathrm{Lip}(\alpha;C)$. Note that one may regard the infinite sum in (4.3) as a discrete form of the convolution integral in (4.11). If $f\in L(\boldsymbol{R})$ is bandlimited to $[-\pi W,\pi W]$, then Parseval's formula states in this regard that

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\pi W}^{\pi W} f^\wedge(v)e^{ivt}dv=W\int_{\boldsymbol{R}} f(u)\frac{\sin\pi W(t-u)}{\pi W(t-u)}du.$$

So (4.11) may be considered as a continuous counterpart of the (discrete) Theorem 4, both being valid for not necessarily bandlimited functions.

5. Round-off Error in Sampling Series. When setting up the sampling sum it may happen that one does not have the exact sample values $f(k/W)$ at one's disposal but only recorded or tabulated values $\bar f(k/W)$, both differing by

$$\varepsilon_k:=f(k/W)-\bar f(k/W),$$

where $|\varepsilon_k|\leq\varepsilon$, $k\in\boldsymbol{Z}$; ε_k is called the local round-off error. In digital signal processing this is the case when the sampled values are replaced by the nearest discrete (quantized) values, namely by the values $\bar f(k/W)$ of the corresponding step function $\bar f$ with possible values $2r\varepsilon$, $r\in\boldsymbol{Z}$ (see Fig. 1). This is assumed below; then $|\varepsilon_k|\leq|f(k/W)|$, $k\in\boldsymbol{Z}$.

In this regard it is of interest to consider the total round-off (or quantization) error

$$(Q_\varepsilon f)(t):=(Q_{\varepsilon,W}f)(t):=f(t)-\sum_{k=-\infty}^{\infty}\bar f\left(\frac{k}{W}\right)\mathrm{si}\{\pi(Wt-k)\}$$

$$=\sum_{k=-\infty}^{\infty}\varepsilon_k\,\mathrm{si}\{\pi(Wt-k)\}\quad(t\in\boldsymbol{R})$$

under the hypotheses of Theorem 1. Round-off errors, possibly caused by uncertainties in the sample values, are generally treated using stochastic methods; see e. g. Ruchkin [36], Papoulis [33] and Ericson [24]. Our main result here, which employs deterministic methods, states

Theorem 5. a) *Let* $f \in C(\mathbf{R}) \cap L(\mathbf{R})$ *with* $f^\wedge(v)=0$ *for all* $|v|>\pi W$, $W>0$. *Then, uniformly in* $t \in \mathbf{R}$, $\lim_{\varepsilon\to 0+}(Q_\varepsilon f)(t)=0$.

b) *In addition assume that* (4.2) *holds for some* $\gamma \in (0, 1]$. *Then* $\|(Q_\varepsilon f)(\cdot)\|_C \leq M_2(f, \gamma)\varepsilon \log(1/\varepsilon)$ *for* $|\varepsilon_k| \leq \varepsilon \leq \min\{1/W, e^{-1/2}\}$, $k \in \mathbf{Z}$, $W \geq 1$, *where* M_2 *is the constant of* (5.3).

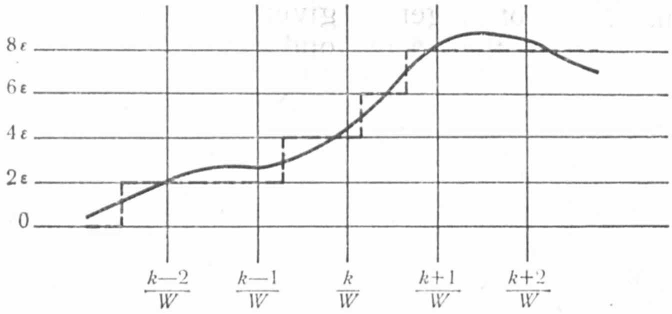

Fig. 1. Function f (drawn as ———) and its corresponding step function $\overline{f}$ (- - - - - -) with values $f(k/W)$ and $\overline{f}(k/W)$, respectively

Concerning the proof of part a), Hölder's inequality gives

$$(5.1) \qquad |(Q_\varepsilon f)(t)| \leq \Big(\sum_{k=-\infty}^{\infty} |\mathrm{si}\{\pi(Wt-k)\}|^q\Big)^{1/q} \cdot \Big(\sum_{k=-\infty}^{\infty} |\varepsilon_k|^p\Big)^{1/p}.$$

Now the first sum is bounded for $q=2$ by 2 according to Lemma 1. Regarding the second sum in (5.1), since $|\varepsilon_k| \leq |f(k/W)|$, $k \in \mathbf{Z}$,

$$\Big(\sum_{k=-\infty}^{\infty} |\varepsilon_k|^2\Big)^{1/2} \leq \Big(\sum_{|k|\leq[1/\varepsilon]} |\varepsilon_k|^2\Big)^{1/2} + \Big(\sum_{|k|>[1/\varepsilon]} |f(\tfrac{k}{W})|^2\Big)^{1/2}.$$

The first term on the right side has upper bound $\sqrt{(2[1/\varepsilon]+1)\varepsilon^2} \leq \sqrt{2\varepsilon+\varepsilon^2}$, which tends to zero for $\varepsilon \to 0+$. The second term also tends to zero in view of the convergence of the series

$$\sum_{k=-\infty}^{\infty} |f(\tfrac{k}{W})|^2 = \int_{-\pi W}^{\pi W} |f^\wedge(v)|^2 dv,$$

valid in view of Parseval's formula since $f^\wedge \in L^2(-\pi W, \pi W)$.

Concerning part b), the second sum in (5.1) is bounded by

$$(5.2) \qquad \Big(\sum_{|k|\leq a(p,\varepsilon)} |\varepsilon_k|^p\Big)^{1/p} + \Big(\sum_{|k|>a(p,\varepsilon)} |f(\tfrac{k}{W})|^p\Big)^{1/p},$$

where $a(p, \varepsilon) := [\varepsilon^{-1/\gamma} W^{p\gamma/(p\gamma-1)}]$. Firstly, note that $a(p, \varepsilon) \geq 1$ provided $\varepsilon \leq 1/W$, $W \geq 1$, and $p\gamma \geq 2$. Then the first term in (5.2), noting that $W^\gamma < 1/\varepsilon$ for $\gamma \in (0, 1]$, is bounded by

$$(\{2a(p,\varepsilon)+1\}\varepsilon^p)^{1/p} \leq 3^{1/p}\varepsilon\varepsilon^{-1/p\gamma}W^{\gamma/(p\gamma-1)} \leq 3^{1/p}\varepsilon \exp\{(4/p\gamma)\log(1/\varepsilon)\} \leq 3^{\gamma/2}\varepsilon e$$

if one chooses $p=(4/\gamma)\log(1/\varepsilon)$, observing that $1/(p\gamma-1) \leq 3/p\gamma$ for $p\gamma \geq 2$, and $3^{1/p} \leq 3^{\gamma/2}$ for $\varepsilon \leq e^{-1/2}$. The second term in (5.2) can then be estimated on account of Lemma 2 by $2^{1/p}M_f W^\gamma a(p,\varepsilon)^{(1-p\gamma)/p} \leq 2^{\gamma/2}M_f \varepsilon e^{1/4}$. Combining all the results, observing that the first sum in (5.1) is less than p, one has

$$(5.3) \qquad |(Q_\varepsilon f)(t)| \leq \frac{4}{\gamma}(3^{\gamma/2}e + 2^{\gamma/2}M_f 2^{1/4})\varepsilon \log \frac{1}{\varepsilon}$$

uniformly in $t \in \boldsymbol{R}$. Note that $p\gamma \geq 2$ as $\varepsilon \leq e^{-1/2}$. This proves part b).

Observe that the convergence given in Theorem 5a) holds uniformly in $t \in \boldsymbol{R}$. That given in the corresponding Theorem 3.1 of [17] holds only for local t-intervals (namely $|t| \leq [1/\varepsilon]/(2W)$). The proof of part b), which is quite different to that of Theorem 3.2 of [17], gets along with less restrictive conditions upon ε. Here $\varepsilon_k = O(|k|^{-\gamma})$, $|k| \to \infty$ in view of (4.2).

It is also possible to examine the total round-off or quantization error in case of sampling approximation of non-bandlimited functions, thus in case of Theorem 2. In comparison with Theorem 5b) it will now be an error additional to that caused by the non-bandlimitation, namely to that of the aliasing error of Theorem 4.

Theorem 6. *Under the hypotheses of Theorem* 4 *one has for* $W = (1/\varepsilon)^{1/(r+\alpha)}$, $r \geq 1$,

$$\| f(\cdot) - \sum_{k=-\infty}^{\infty} \overline{f}\left(\frac{k}{W}\right) \mathrm{si}\{\pi(W\cdot - k)\} \|_C \leq M_3(f, r, \alpha, \gamma)\, \varepsilon \log(1/\varepsilon)$$

for $\varepsilon \leq \exp\{-2(r+\alpha)/(r+\alpha+\gamma)\}$, *where* $M_3 := M_1 + M_2$, M_1 *and* M_2 *being the constants in* (4.7) *and* (5.3).

To prove it, one splits up the term within the norm as

$$\{f(t) - \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \mathrm{si}\{\pi(Wt-k)\}\} + \{\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{W}\right) - \overline{f}\left(\frac{k}{W}\right)\right) \mathrm{si}\{\pi(Wt-k)\}\}.$$

The norm of the first term is of order $O(\log W/W^{r+\alpha})$ by Theorem 4, that of the second term of order $O(\varepsilon \log 1/\varepsilon)$ by Theorem 5b). Taking W as postulated, the result follows.

Note that it can be shown that Theorem 6 also holds for $r=0$ by modifying the proof of Theorem 5b) slightly. It is not to be expected that the rate of convergence in Theorem 6 can be any better than $O(\varepsilon \log 1/\varepsilon)$, even if higher derivatives of f exist. However, in this situation the sampling rate can be diminished without increasing the error. Indeed, the existence of the $(r+2)$-th continuous derivative instead of just the $(r+1)$-th allows one to multiply the distance between the sampling nodes by the factor $\exp(-1/(r+r^2))$. For example, for $r=1$ and $\varepsilon = 1/4$ this will be the factor 2.

The assertions of Theorems 5b) and 6 may be interpreted in terms of stability theory with rates: a small change in the function values at all of the nodes produces a corresponding small change in the sampling expansion on the entire $\boldsymbol{R}$.

Note that Theorem 6 holds in particular for duration limited functions.

6. Time Jitter Error in the Sampling Theorem. When trying to set up the sampling sums it may also happen that the samples cannot be taken at the instants k/W but at $(k/W + \delta_k)$, the sampled values now being $f(k/W + \delta_k)$. These errors in timing give rise to the jitter error

$$(J_\delta f)(t) := (J_{\delta,W} f)(t) := f(t) - \sum_{k=-\infty}^{\infty} f(\frac{k}{W}+\delta_k)\,\mathrm{si}\,\{\pi(Wt-k)\}$$

$$= \sum_{k=-\infty}^{\infty} (f(\frac{k}{W}) - f(\frac{k}{W}+\delta_k))\,\mathrm{si}\,\{\pi(Wt-k)\} \quad (t \in \boldsymbol{R})$$

under the hypotheses of Theorem 1.

The calculation of this error has so far been carried out using stochastic methods — the δ_k being regarded as a weak sense stationary discrete-parameter random process having finite variance; see e. g. Balakrishnan [1], Brown and Palermo [9], Beutler and Leneman [3] and Beutler [2]. It will here be treated using strictly deterministic methods, based on the sole assumption that $|\delta_k| \leq \delta$ for $k \in \boldsymbol{Z}$, $\delta > 0$ sufficiently small.

Theorem 7. *Let* $f \in C(\boldsymbol{R}) \cap L(\boldsymbol{R})$ *satisfy* (4.2) *for* $0 < \gamma \leq 1$ *with* $f^\wedge(v) = 0$ *for all* $|v| > \pi W$, $W > 0$. *Then*

$$\|(J_\delta f)(\cdot)\|_C \leq M_4(f, f', \gamma)\delta \log(1/\delta)$$

provided $|\delta_k| \leq \delta \leq \min\{1/W, e^{-1/2}\}$, $k \in \boldsymbol{Z}$, $W \geq 1$, *where* M_4 *is the constant of* (6.3).

Regarding the proof, by Hölder's inequality

$$(6.1) \qquad |(J_\delta f)(t)| \leq (\sum_{k=-\infty}^{\infty} |\mathrm{si}\,\{\pi(Wt-k)\}|^q)^{1/q} (\sum_{k=-\infty}^{\infty} |f(\frac{k}{W}) - f(\frac{k}{W}+\delta_k)|^p)^{1/q}$$

for $1/p + 1/q = 1$. The second sum is bounded by

$$(6.2) \quad (2b+3)^{1/p} \sup_{k \in \boldsymbol{Z}} \|f(\cdot) - f(\cdot + \delta_k)\|_C + (\sum_{|k| \geq b+2} |f(\frac{k}{W}) - f(\frac{k}{W}+\delta_k)|^p)^{1/p},$$

where $b = b(p, \delta) := [\delta^{-1/\gamma} W^{p\gamma/(p\gamma-1)}]$. Again $b \geq 1$ provided $\delta \leq 1/W$, $W \geq 1$ and $p\gamma \geq 2$. The first term in (6.2) is bounded by $(2b+3)^{1/p}\delta \|f'\|_C$, noting that $f' \in C(\boldsymbol{R})$ by the hypotheses. Then, since $W^\gamma < 1/\delta$ for $0 < \gamma \leq 1$, $(2b+3)^{1/p} \leq (5b)^{1/p} \leq 5^{1/p} \exp\{(4/p\gamma)\log(1/\delta)\} \leq 5^{\gamma/2} e$ if one chooses $p = (4/\gamma)\log(1/\delta)$ and takes $\delta \leq e^{-1/2}$. The second term in (6.2) is now bounded by Lemma 2 by $2 \cdot 2^{1/p} M_f W^\gamma b^{(1-p\gamma)/p\gamma} \leq 2 \cdot 2^{\gamma/2} M_f \delta e^{1/4}$. Since the first sum in (6.1) is bounded by p, a combination of all estimates yields

$$(6.3) \qquad |(J_\delta f)(t)| \leq \frac{4}{\gamma} \{5^{\gamma/2} e \|f'\|_C + 2 \cdot 2^{\gamma/2} M_f e^{1/4}\} \delta \log(\frac{1}{\delta}).$$

The foregoing theorem seems to be new. Deterministic methods had previously been used in Butzer and Splettstösser [18] to study the jitter error for generalized sampling sums which are discretizations of convolution integrals on $\boldsymbol{R}$.

It is also feasible to study the jitter error in the case of not necessarily bandlimited functions.

Theorem 8. *Under the hypotheses of Theorem* 4 *one has for* $W = (1/\delta)^{1/(r+\alpha)}$, $r \geq 1$,

$$\|f(\cdot) - \sum_{k=-\infty}^{\infty} f(\frac{k}{W}+\delta_k)\,\mathrm{si}\,\{\pi(W\cdot - k)\}\|_C \leq M_5(f, f', r, \alpha, \gamma)\delta \log(1/\delta)$$

for $\delta \leq \exp\{-2(r+\alpha)/(r+\alpha+\gamma)\}$, *where* $M_5:=M_1+M_4$, M_1 *and* M_4 *being the constants in* (4.7) *and* (6.3).

For the proof, split up the term in the norm as

$$\{f(t)-\sum_{k=-\infty}^{\infty} f(\tfrac{k}{W})\,\mathrm{si}\,\{\pi(Wt-k)\}\}+\sum_{k=-\infty}^{\infty}(f(\tfrac{k}{W})-f(\tfrac{k}{W}+\delta_k))\,\mathrm{si}\,\{\pi(Wt-k)\}.$$

Whereas the first term is of order $O(\log W/W^{r+\alpha})$ by Theorem 4, that of the second is $o(\delta \log 1/\delta)$ by Theorem 7.

According to Theorems 7 or 8, the sampling expansion also exemplifies stability with respect to the nodal values: a uniformly small error in each of the nodal values produces a correspondingly small error in the recovered signal.

7. Sampling Theorem for Weak Sense Stationary Stochastic Processes. Since signal functions are often of random character, random signals play an important role in signal processing and in the sampling theorem. For this purpose one usually uses stochastic processes which are stationary in the weak sense as a model for them. Given a probability space $(\Omega, \mathscr{A}, P)$, a stochastic process, namely an $\mathscr{A}$-measurable function $X=X(t)=X(t,\omega)$ of $\omega \in \boldsymbol{R}$ for each $t \in \boldsymbol{R}^+$, is said to be weak sense stationary (w. s. s.), if its autocorrelation function (a. c. f.)

$$A_{X,X}(t, t+\tau):=\int_\Omega X(t,\omega)X(t+\tau,\omega)\,dP(\omega)$$

is independent of $t \in \boldsymbol{R}$, i. e., $A_{X,X}(t,t+\tau)=A_{X,X}(\tau):=A_X(\tau)$. Here it is assumed that X is square integrable with respect to P over Ω, i. e., $E\{|X(t)|^2\}:=\int_\Omega |X(t,\omega)|^2 dP(\omega)<\infty$ $(t \in \boldsymbol{R})$. Such a w. s. s. process X is said to be bandlimited to the interval $[-\pi W, \pi W]$ if the deterministic a.c.f. A_X is bandlimited there. We shall state the sampling theorem for such processes and then examine the time jitter error in more detail.

Theorem 9. *Let X be a* w.s.s. *process with* $E\{|X(t)|^2\}<\infty$, $t \in \boldsymbol{R}$' *such that X is bandlimited to* $[-\pi W, \pi W]$. *Then*

a) $$\lim_{N\to\infty} E\{|X(t,\omega)-\sum_{k=-N}^{N} X(\tfrac{k}{W},\omega)\,\mathrm{si}\,\{\pi(Wt-k)\}|^2\}=0.$$

b) *If in addition the* a. c. f. A_X *satisfies* (4.2) *for some* $\gamma \in (0,1]$ *then**

(7.1) $$|E\{|X(t,\omega)-\sum_{k=-\infty}^{\infty} X(\tfrac{k}{W}+\delta_k,\omega)\,\mathrm{si}\,\{\pi(Wt-k)\}|^2\}|^{1/2} \leq M_6(A, A_X'', \gamma)\delta \log(1/\delta)$$

for $|\delta_k| \leq \delta \leq \min\{1/W, e^{-2/5}\}$, $k \in \boldsymbol{Z}$, $W \geq 1$, *where* M_6 *is the square root of the constant in* (7.6).

Concerning the proof of part a) see [49]. Regarding part b), since X is bandlimited, one can rewrite the square of the left-hand side of (7.1) by part a) as

* Here $E\{|f(t)-\Sigma_{k=-\infty}^{\infty} g_k(t)|^2\}$ is to be understood as $\lim_{N\to\infty} E\{|f(t)-\Sigma_{k=-N}^{N} g_k(t)|^2\}$.

$$(7.2)\qquad \int_\Omega \Big| \sum_{k=-\infty}^{\infty} \Big(X\Big(\frac{k}{W}\Big) - X\Big(\frac{k}{W}+\delta_k\Big)\Big) \operatorname{si}\{\pi(Wt-k)\}\Big|^2 dP(\omega)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \Delta_{-\delta_j}(\Delta_{\delta_k} A_X)\Big(\frac{k-j}{W}\Big) \operatorname{si}\{\pi(Wt-j)\} \operatorname{si}\{\pi(Wt-k)\}$$

where $\Delta_{\pm\delta_k} A_X(l/W) := A_X(l/W \pm \delta_k) - A_X(l/W)$ $(k, l \in \boldsymbol{Z})$. Under the notation $D(l/W;\ \delta) := \sup\{|\Delta_{\delta'}(\Delta_{\delta''} A_X)(l/W)| : |\delta'| \le \delta, |\delta''| \le \delta\}$ $(l \in \boldsymbol{Z})$ (7.2) is bounded by

$$(7.3)\qquad \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} D\Big(\frac{k-j}{W};\ \delta\Big) | \operatorname{si}\{\pi(Wt-j)\} \operatorname{si}\{\pi(Wt-k)\}|$$

$$\le \Big\{\sum_{k=-\infty}^{\infty} |\operatorname{si}\{\pi(Wt-k)\}|^q\Big\}^{1/q} \Big\{\sum_{k=-\infty}^{\infty} \Big| \sum_{j=-\infty}^{\infty} D\Big(\frac{k-j}{W};\ \delta\Big) | \operatorname{si}\{\pi(Wt-j)\}|\Big|^p\Big\}^{1/p}$$

in view of Hölder's inequality with $1/p + 1/q = 1$. Now the latter double sum is bounded on account of the Hausdorff-Young inequality by

$$(7.4)\qquad \Big\{\sum_{j=-\infty}^{\infty} |\operatorname{si}\{\pi(Wt-j)\}|^s\Big\}^{1/s} \Big\{\sum_{j=-\infty}^{\infty} D\Big(\frac{j}{W};\ \delta\Big)^r\Big\}^{1/r},$$

where $0 \le 1/s + 1/r - 1 = 1/p$. The right-hand sum in this product is bounded by, noting that $A_X''(t) \in C(\boldsymbol{R})$,

$$(7.5)\qquad (2c(r, \delta) + 5)^{1/r} \| A_X'' \|_C \delta^2 + \Big\{\sum_{|j| > c+2} D\Big(\frac{j}{W};\ \delta\Big)^r\Big\}^{1/r},$$

where $c = c(r, \delta) := [\delta^{-2/\gamma} W^{r\gamma/(r\gamma-1)}]$. First note that $c(r, \delta) \ge 2$ provided $\delta \le e^{-2/5}$, $W \ge 1$ and $r\gamma \ge 2$. Indeed, $c \ge [e^{4/5} W^2] \ge [2.225] = 2$. Now noting that $W \le (1/\delta)$ for $\gamma \in (0, 1]$,

$$(2c+5)^{1/r} < (9c/2)^{1/r} < (9/2)^{1/r} \delta^{-2/r\gamma} W^{\gamma/(r\gamma-1)}$$

$$\le (9/2)^{1/r} \exp\Big\{\Big(\frac{2}{\gamma r} + \frac{1}{r\gamma - 1}\Big) \log(1/\delta)\Big\} \le (9/2)^{1/r} \exp\Big\{\frac{5}{\gamma r} \log(1/\delta)\Big\} \le 2^{\gamma/2}(3/2)^\gamma e$$

if one chooses $r = (5/\gamma)\log(1/\delta)$. The second term in (7.5) is then bounded according to Lemma 2 by

$$2^{1/r} 4 M_A W^\gamma c(r, \delta)^{(1-4\gamma)/r} \le 2^{\gamma/2} 4 M_A \delta^2 e^{2/5}.$$

Setting $s = 2r/(2r-1)$, then $p = 2r > 0$, and the left-hand sum in (7.4) is bounded by $(s')^{1/s} \le s' = 2r$, where $1/s + 1/s' = 1$. Likewise the first sum on the right in (7.3) is bounded by $p = 2r$. Combining all the results yields that (7.2) is bounded by

$$(7.6)\qquad \frac{2^{\gamma/2} 100}{\gamma^2} \{(3/2)^\gamma e \| A_X'' \|_C + 4 e^{2/5} M_A\} (\delta \log(1/\delta))^2$$

uniformly in $t \in \boldsymbol{R}$. This completes the proof.

Part b) of the foregoing theorem seems to be new. It would be of interest to study the counterpart of part b) for duration limited w. s. s. processes or for not necessarily bandlimited ones. The quantization error for such processes should be a problem of further interest. The methods employed in this paper should be sufficiently powerful to handle the matter. On the other hand, the counterpart of Theorem 4, namely the aliasing error for w. s. s. processes, as well as other interesting generalizations, have already been studied in detail by Splettstösser [46, 49].

General linear stochastic processes which are not necessarily stationary (in the strict or weak sense), nor have independent increments, have been treated in Butzer and Gather [13], however from the point of view of the central limit theorem with rates. These general processes include some random noise as well as pulse train processes as specific models.

8. Further Extensions. It is possible to extend the sampling theorem and the various problems connected with it in many more ways than has been carried out above. Let me indicate just a few of these.

(i) Sampling expansions involving sampled values of the function as well as of its derivative, thus simple Hermite-type interpolation: if f, $f' \in C(R) \cap L(R)$ and $[f']^{\wedge} \in L(R)$, then, uniformly in $t \in R$,

$$f(t) = \lim_{W\to\infty} \sum_{k=-\infty}^{\infty} \frac{\{\sin(\pi/2)(Wt-k)\}^2}{(\pi/2)(wt-k)} \{f(\tfrac{2k}{W}) + (t - \tfrac{2k}{W}) f'(\tfrac{2k}{W})\}.$$

In case of bandlimited functions the limit is dropped. Note that the nodes above are double the distance apart compared with that in (2.1). See [16] for the associated error estimates and the literature.

(ii) The Hilbert transform, defined by $f^{\sim}(t) = \lim_{\delta\to 0}(1/\pi)\int_{|u|\geqq\delta}(f(t-u)/u)du$, can be approximated using samples of f: if $f \in C(R) \cap L(R)$, $f^{\wedge} \in L(R)$ and $f^{\sim} \in C(R)$, then (see [16])

$$f^{\sim}(t) = \lim_{W\to\infty} \sum_{k=-\infty}^{\infty} f(\tfrac{k}{W}) \frac{\{\sin(\pi/2)(Wt-k)\}^2}{(\pi/2)(Wt-k)}.$$

For rates in this regard see [50].

(iii) Reconstruction by generalized sampling sums: the function $(\sin t)/t$ in (1.1) is replaced by $g \in C(R) \cap L(R)$ such that $(1/\sqrt{2\pi})\int_{\boldsymbol{R}} g(u)du = 1$, $g^{\wedge}(v) = 0$ for $|v| > V$, some $V > 0$. Then for each $f \in C_0(R)$ and uniformly in $t \in R$,

$$f(t) = \lim_{W\to\infty} \frac{\sqrt{(\pi/2)}}{V} \sum_{k=-\infty}^{\infty} f(\tfrac{k}{W})\, g(\tfrac{\pi}{V}(Wt-k)). \tag{8.1}$$

A concrete example of a function g satisfying these hypotheses is the kernel function of de La Vallée Poussin in (4.4), namely $g(t) := (3/\sqrt{2\pi}) \times \mathrm{si}\{3t/2\}\,\mathrm{si}\{t/2\}$ for which $V = 2$. In this case the rate of approximation in (8.1) is of order $O(1/W^{r+\alpha})$, $r \in P$, $0 < \alpha < 1$ if and only if $f^{(r)} \in \mathrm{Lip}(\alpha, C)$. If $\alpha = 1$, the saturation case, the Lipschitz class has to be replaced by the Zygmund class. See Stens [53]. A further example is the Fejér kernel $g(t) := 1/\sqrt{2\pi}(\mathrm{si}\{t/2\})^2$ with $V = 1$. For each $f \in C_0(R)$, uniformly in $t \in R$,

$$f(t) = \lim_{W\to\infty} \frac{1}{2} \sum_{k=-\infty}^{\infty} f(\tfrac{k}{W})(\mathrm{si}\{\tfrac{\pi}{2}(Wt-k)\})^2.$$

Note that the sum in (8.1) may be regarded as a discretized convolution sum of the associated convolution integral of f and $(\pi W/V)g(\pi Wt/V)$, namely of

$$\frac{1}{\sqrt{2\pi}} \int_R f(u) \frac{\pi W}{V} g\left(\frac{\pi W}{V}(t-u)\right) du \quad (t \in R).$$

So well-known results on singular convolution integrals ([14]) can be applied. For further results in this direction see [43, 44], [52, 53].

(iv) The truncation error, resulting when only a finite number of samples (namely $2N+1$) are used for the representation, has been studied very intensively (see [27]). If $f \in C(R) \cap L(R)$ is bandlimited to $[-\pi W, \pi W]$ such that $[f^\wedge]^{(r)} \in \text{Lip}(\alpha, C)$, then

$$\| f(\cdot) - \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) \text{si}\{\pi(W\cdot - k)\} \|_C = O(N^{-r-\alpha}) \quad (N \to \infty).$$

This estimate is one of many to be found in Scheben [37]. A typical result in the non-bandlimited case states: If $f \in C(R) \cap L(R)$ satisfies (4.2) for $\gamma \in (0, 1]$, $f^{(r)} \in \text{Lip}(\alpha, C)$, and $N = [W^{2(1+(r+\alpha)/\gamma)}] + 1$, then

$$\| f(\cdot) - \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) \text{si}\{\pi(W\cdot - k)\} \|_C = O(W^{-r-\alpha} \log W) \quad (W \to \infty).$$

See [17], and for a different approach Honda [26].

(v) Replacement of the trigonometric system by general orthogonal systems on a finite or infinite interval; see Kramer [29] and the many papers by Jerri cited in [27], respectively. In particular, the sampling theorem for the Walsh system, both for sequency-limited and duration-limited functions, is considered e. g. in Maqusi [31, 17, 45], Engels and Splettstösser [23]. Comparable results for the Haar system are to be found in Ziegler [60]. For the sampling theorem in the Legendre frame on R^+ together with a new type of truncation error estimate see Butzer, Stens and Wehrens [19].

(vi) Sampling theorems for functions with multi-dimensional domain, basic for picture processing and transmission. For such results in case of the classical trigonometric system see Splettstösser [47] and the extensive literature cited there. For the multi-dimensional Walsh setting see Butzer and Engels [12].

Acknowledgement. It is a pleasure to take this opportunity to thank the many members of the "DFG Schwerpunktprogramm Digitale Signalverarbeitung" (Bu 166/29/31/33/36/38), in particular Professors A. Fettweis (Bochum), H. D. Lüke (Aachen), W. Schüssler (Erlangen), H. J. Tafel (Aachen), D. Wolf (Frankfurt) for many stimulating discussions and suggestions during the various meetings of this research group in the course of the past four years which were coordinated in an expert fashion by Dipl. Ing K. Seifert (DFG, Bonn-Bad Godesberg). A good part of the problems treated in the overview were posed there. Professor O. Lange (Aachen) kindled the author's first interest in the sampling theorem.

The author would like to extend his special thanks to his research companion Dr. W. Splettstösser (Aachen) for his unstinting help when writing up the manuscript. The proofs especially of Theorems 5, 7 and 9 benefited greatly from his expert advice. Dr. R. L. Stens (Aachen) read the final version with meticulous care and gave valuable suggestions. He aided with the proof of Theorem 2. Finally, the author wishes to thank the DFG for their continued support, also making it possible for him to attend the Varna conference.

REFERENCES

1. A. V. Balakrishnan. On the problem of time jitter in sampling. *IRE Trans. Inform. Theory*, **IT-8**, 1962, 226-236.
2. F. J. Beutler. Recovery of randomly sampled signals by simple interpolators. *Inform. and Control*, **26**, 1974, 312-340.
3. F. J. Beutler, O. A. Z. Leneman. Random sampling of random processes. *Inform. and Control*, **9**, 1966, 325-246.
4. R. P. Boas. Entire Functions. New York, 1954, p. 220.
5. R. P. Boas. Summation formulas and band-limited signals. *Tôhoku Math. J.*, **24**, 1972, 121-125.
6. R. P. Boas, H. Pollard. Continuous analogues of series. *Amer. Math. Monthly*, **80**, 1973, 18-25.
7. J. L. Brown. On the error in reconstructing a non-bandlimited function by means of the bandpass sampling theorem. *J. Math. Anal. Appl.*, **18**, 1967, 75-84.
8. J. L. Brown. Sampling theorem for finite energy signals. *IEEE Trans. Inform. Theory*, **IT-14**, 1968, 818-819.
9. W. M. Brown, C. J. Palermo. System performance in the presence of stochastic delays. *IRE Trans. Inform. Theory*, **IT-8**, 1962, 206-214.
10. V. I. Buslaev, A. G. Vituškin. An estimate of the length of a signal code with a finite spectrum in connection with sound transcription problems. *Izv. Akad. Nauk USSR, Ser. Mat.*, **38**, 1975, 867-895.
11. P. L. Butzer. The Banach-Steinhaus theorem with rates, and applications to various branches of analysis. — In: General Inequalities. II. Basel, 1980, 299-331.
12. P. L. Butzer, W. Engels. Dyadic calculus and sampling theorems for functions with multi-dimensional domain. Part I: General theory. Part II: Applications to dyadic sampling representation (to appear).
13. P. L. Butzer, U. Gather. Asymptotic expansions for central limit theorems for general linear stochastic processes. I. General theorems on rates of convergence. II. Models of the general random noise and pulse train processes. *Math. Methods Appl. Sci.*, **1**, 1979, 241-264, 346-353.
14. P. L. Butzer, R. J. Nessel. Fourier Analysis and Approximation. Basel, 1971.
15. P. L. Butzer, W. Splettstösser. A sampling theorem for duration-limited functions with error estimates. *Inform. and Control*, **34**, 1977, 55-65.
16. P. L. Butzer, W. Splettstösser. Approximation und Interpolation durch verallgemeinerte Abtastsummen. *Forsch. Ber.*, NRW 2708, Opladen, 1977, p. 57.
17. P. L. Butzer, W. Splettstösser. Sampling principle for duration-limited signals and dyadic Walsh analysis. *Inform. Sci.*, **14**, 1978, 93-106.
18. P. L. Butzer, W. Splettstösser. On quantization, truncation and jitter errors in the sampling theorem and its generalizations. *Signal Process*, **2**, 1980, 101-112.
19. P. L. Butzer, R. L. Stens, M. Wehrens. The continuous Legendre transform, its inverse transform, and applications. *Internat. J. Math. & Math. Sci.*, **3**, 1980, 47-67.
20. L. L. Campbell. A comparison of the sampling theorems of Kramer and Whittaker. *SIAM J.*, **12**, 1964, 117-130.
21. E. W. Cheney. Introduction to Approximation Theory. New York, 1966.
22. H. Dym, H. P. McKean. Fourier Series and Integrals. New York, 1972.
23. W. Engels, W. Splettstösser. A note on Maqusi's proofs and truncation error bounds for the dyadic (Walsh) sampling theorem. *IEEE Trans. Acoust., Speech, Signal Proc.* (to appear).
24. T. Ericson. A generalized version of the sampling theorem. *Proc. IEEE*, **60**, 1972, 1554-1555.
25. W. L. Ferrar. On the cardinal function of interpolation-theory. *Proc. Roy. Soc. Edinburgh*, **46**, 1925, 323-333.
26. I. Honda. Approximation for a class of signal functions by sampling series representation. *Keio Engineering Reports*, **31**, 1978, 21-26.
27. A. J. Jerri. The Shannon sampling theorem — its various extensions and applications: a tutorial review. *Proc. IEEE*, **65**, 1977, 1565-1596.
28. V. A. Kotelnikov. On the truncation capacity of "ether" and wire in electrocommunications (Russian). — In: Material for the First All-Union Conference on Questions of Communications. Moscow, 1933.

29. H. P. Kramer. A generalized sampling theorem. *J. Math. Phys.*, **38**, 1959, 68-72.
30. H. D. Lüke. Zur Entstehung des Abtasttheorems. *Nachr. Techn. Z.*, **31**, 1978, 271-274.
31. M. Maqusi. Sampling representation of sequency-band-limited nonstationary random processes. *IEEE Trans. Acoust., Speech, Signal Proc.*, ASSP-**28**, 1980, 249-251.
32. J. McNamee, F. Stenger, E. L. Whitney. Whittaker's cardinal function in retrospect. *Math. Comp.*, **25**, 1971, 141-154.
33. A. Papoulis. Error analysis in sampling theory. *Proc. IEEE*, **54**, 1966, 947-955.
34. H. Pollard, O. Shisha. Variations on the binomial series. *Amer. Math. Monthly*, **79**, 1972, 495-499.
35. H. Raabe. Untersuchungen an der wechselseitigen Mehrfachübertragung (Multiplexübertragung). *Elektr. Nachrichtentechnik*, **16**, 1939, 213-228.
36. D. S. Ruchkin. Linear reconstruction of quantized and sampled random signals. *IRE Trans. Commun. Syst.*, CS **9**, 1961, 350-355.
37. U. Scheben. Über den Abbruchfehler bei Abtastreihen. Diplomarbeit. Aachen, 1980.
38. A. Schönhage. Approximationstheorie. Berlin, 1971.
39. C. E. Shannon. Communication in the presence of noise. *Proc. IRE*, **37**, 1949, 10-21.
40. L. B. Sofman. A problem of Shannon. *Dokl. Akad. Nauk USSR*, **215**, 1974, 1313-1316.
41. I. Someya. Waveform Transmission. Tokyo, 1949.
42. W. Splettstösser. Some extensions of the sampling theorem. — In: Linear Spaces and Approximation. ISNM. Vol. 40. Basel, 1978, 615-628.
43. W. Splettstösser. On generalized sampling sums based on convolution integrals. *Arch. Elek. Übertr.*, **32**, 1978, 267-275.
44. W. Splettstösser. Error estimates for sampling approximation of non-bandlimited functions. *Math. Methods Appl. Sci.*, **1**, 1979, 127-137.
45. W. Splettstösser. Error analysis in the Walsh sampling theorem. — In: 1980 IEEE Int. Symp. on Electromagnetic Compatibility, Baltimore. Inst. for Electrical & Electronics Engineers, Service Center, Piscataway. New Jersey, 366-370.
46. W. Splettstösser. On the approximation of random processes by convolution processes. *Z. Angew. Math. Mech.*, **61**, 1981, 235-241.
47. W. Splettstösser. Bandbegrenzte und effektiv bandbegrenzte Funktionen und ihre Praediktion aus Abtastwerten. (Habilitationsschrift, Aachen, 1981, 65 pp.)
48. W. Splettstösser. Sampling approximation of continuous functions with multidimensional domain. *IEEE Trans. Inform. Theory* (in print).
49. W. Splettstösser. Sampling series approximation of continuous weak sense stationary processes (to appear).
50. W. Splettstösser, R. L. Stens, G. Wilmes. On approximation by the interpolating series of G. Valiron. *Funct. Approx. Comment. Math.*. **11**, 1980, 39-56.
51. F. Stenger. Approximations via Whittaker's cardinal function. *J. Approx. Theory* **17**, 1976, 222-240.
52. R. L. Stens. Approximation to duration-limited functions by sampling sums. *Signal Process.*, **2**, 1980, 173-176.
53. R. L. Stens. Error estimates for sampling sums based on convolution integrals. *Inform. and Control*, **45**, 1980, 37-47.
54. H. Triebel. Fourier Analysis and Function Spaces. Leipzig, 1977, p. 30, 48 f.
55. C. J. de La Vallée Poussin. Sur la convergence des formules d'interpolation entre ordonnées équidistantes. *Bull. Acad. Roy. de Belgique*, 1908, 319-410.
56. A. G. Vituškin. Approximations of entire functions and coding of signals with finite spectrum. — In: Approximation Theory. II. New York, 1976, 269-281.
57. E. T. Whittaker. On the functions which are represented by the expansions of the interpolation-theory. *Proc. Roy. Soc. Edinburgh.*, **35**, 1915, 181-194.
58. J. M. Whittaker. The Fourier theory of the cardinal functions. *Proc. Math. Soc. Edinburgh*, **1**, 1929, 169-176.
59. J. M. Whittaker. Interpolatory Function Theory. Cambridge, 1935.
60. W. Ziegler. Haar-Fourier Transformation auf dem R^+. Dissertation, Technische Universität München, 1981.
61. A. Zygmund. Trigonometric Series. Reprint of Second edition. Cambridge, 1968.

Rhein.-Westf. Technische Hochschule
Lehrstuhl A für Mathematik
51 Aachen *BRD*

Received on July 14, 1981