КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'81
София, 1983, с. 275—278

CONSTRUCTIVE FUNCTION THEORY'81
Sofia, 1983, p. 275—278

ON CONDENSATION PRINCIPLES WITH RATES. II

W. Dickmeis*, R. J. Nessel

Summary. This note is concerned with an extension of the classical condensation principle (Banach-Steinhaus (1927)) to a version with rates for not necessarily countable index sets. Again the proof proceeds via a suitable modification of the familiar gliding hump method. An application is given to the approximate calculation of indefinite integrals.

Let X be a Banach space, Y a normed linear space, and $U \subset X$ a linear manifold with seminorm $|\cdot|_U$. As usual the K-functional $(t \geqq 0)$

(1) $$K(t, f) := K(t, f;\ X,\ U) := \inf\{\| f-g\|_X + t\,|g|_U :\ g \in U\}$$

serves as an abstract measure of smoothness for $f \in X$ (cf. (19)). Let $\{\{T_{n,\lambda}\}_{n\in N};\ \lambda \in \Lambda\}$ be a family of sequences of bounded linear operators of X into Y (i. e., $T_{n,\lambda} \in [X, Y]$), where N is the set of natural numbers and Λ an arbitrary index set. Let $\{\varphi_n\}_{n\in N}$ denote a (strictly) positive sequence with

(2) $$\lim_{n\to\infty} \varphi_n = 0.$$

As a consequence of a familiar direct approximation (or Jackson-type) theorem (Butzer-Scherer (1972)) one has

Proposition. *If for a constant $M \geqq 0$, independent of $n \in N$, $\lambda \in \Lambda$,*

(3) $$\| T_{n,\lambda}\|_{[X,Y]} \leqq M,$$

(4) $$\| T_{n,\lambda} f\|_Y \leqq M\varphi_n |f|_U \quad (f \in U),$$

then one has uniformly for all $n \in N$, $\lambda \in \Lambda$

(5) $$\| T_{n,\lambda} f\|_Y \leqq MK(\varphi_n, f) \quad (f \in X).$$

Indeed, for any $f \in X$, $g \in U$

$$\| T_{n,\lambda} f\|_Y \leqq \| T_{n,\lambda}(f-g)\|_Y + \| T_{n,\lambda} g\|_Y \leqq M(\| f-g\|_X + \varphi_n |g|_U),$$

so that (5) follows upon taking the infimum over all $g \in U$.

* Supported in part by DFG, Grant Ne 171/5-1.

The problem to be discussed in this note is whether the rates given via (5) are indeed sharp. To this end, let ω be a modulus of continuity, thus a continuous, increasing function on $[0, \infty)$ satisfying

$$\omega(0)=0, \quad \omega(t)>0 \quad \text{for } t>0, \tag{6}$$

$$\omega(t+s)\leqq\omega(t)+\omega(s).$$

Note that for each modulus of continuity one has

$$\omega(t)/t\leqq 2\omega(s)/s \quad (t\geqq s>0). \tag{7}$$

For elements of the intermediate classes $U\subset X_\omega\subset X$, $X_\omega :=\{f\in X;\ K(t,f) =O_f(\omega(t)),\ t\to 0+\}$, assertion (5) then delivers the rate (uniformly for $\lambda\in\Lambda$)

$$\| T_{n,\lambda} f \|_Y = O_f(\omega(\varphi_n)) \quad (f\in X_\omega). \tag{8}$$

Excluding the limiting case $\omega(t)=O(t)$, i. e., imposing

$$\lim_{t\to 0+} \omega(t)/t=\infty, \tag{9}$$

one has the following result concerning the sharpness of (8):

Theorem. *Let the spaces X, Y, U, X_ω be given as above, and $\{\varphi_n\}$, ω satisfy* (2), (6), (9), *respectively. Let $\{\{T_{n,\lambda}\}_{n\in N};\ \lambda\in\Lambda\}\subset[X, Y]$ possess the properties* (3, 4). *Moreover, let there exist constants $C_{1,2,3}>0$, independent of $n\in \mathbf{N}$, $\lambda\in\Lambda$, and elements $h_n\in U$ such that*

$$\| h_n \|_X\leqq C_1, \tag{10}$$

$$| h_n |_U\leqq C_2/\varphi_n, \tag{11}$$

$$\liminf_{n\to\infty} \| T_{n,\lambda} h_n \|_Y\geqq C_3 \tag{12}$$

for each $\lambda\in\Lambda$. Then there exists $f_\omega\in X_\omega$ such that

$$\limsup_{n\to\infty} \| T_{n,\lambda} f_\omega \|_Y/\omega(\varphi_n)\geqq C_3/6 \tag{13}$$

simultaneously for all $\lambda\in\Lambda$.

Proof. Starting with $n_1=1$, one may successively construct a subsequence $\{n_k\}_{k\in N}\subset \mathbf{N}$ such that for $k\geqq 2$

$$n_k>n_{k-1}, \quad \varphi_{n_k}<\varphi_{n_{k-1}}, \tag{14}$$

$$\omega(\varphi_{n_k})<r\omega(\varphi_{n_{k-1}}), \quad r\leqq\min\{1/2,\ C_3/6MC_1\}, \tag{15}$$

$$\sum_{j=1}^{k-1} \omega(\varphi_{n_j})/\varphi_{n_j}\leqq\omega(\varphi_{n_k})/\varphi_{n_k}, \tag{16}$$

$$\| T_{n_k,\lambda} g_{k-1} \|_Y\leqq(C_3/3)\omega(\varphi_{n_k}), \tag{17}$$

$$g_{k-1} := \sum_{j=1}^{k-1} \omega(\varphi_{n_j}) h_{n_j}\in U.$$

Indeed, if the first $k-1$ elements $n_1, \ldots, n_{k-1}$ and thus g_{k-1} are given' one may take n_k large enough to satisfy (14)—(16) in view of (2), (6), (9)' Moreover, (17) may be satisfied because of (4) since $g_{k-1} \in U$ and $\varphi_n = o(\omega(\varphi_n))$ (cf. (9)). Since X is complete and (cf. (6), (10), (15))

$$\text{(18)} \qquad \sum_{j=k+1}^{\infty} \omega(\varphi_{n_j}) \| h_{n_j} \|_X \leqq C_1 \omega(\varphi_{n_{k+1}}) \sum_{j=0}^{\infty} r^j \leqq 2C_1 \omega(\varphi_{n_{k+1}}),$$

the element $f_\omega := \Sigma_{j=1}^{\infty} \omega(\varphi_{n_j}) h_{n_j}$ is well defined in X. Moreover, $f_\omega \in X_\omega$. In fact, since for each $t \in (0, \varphi_{n_1})$ there exists k such that $\varphi_{n_{k+1}} \leqq t < \varphi_{n_k}$ (cf. (2), (14)), one obtains by (1), (11), (16), (18) and finally by (7) that

$$K(t, f_\omega) \leqq \| f_\omega - g_k \|_X + t \,| g_k |_U \leqq \sum_{j=k+1}^{\infty} \omega(\varphi_{n_j}) \| h_{n_j} \|_X + t \sum_{j=1}^{k} \omega(\varphi_{n_j}) \,| h_{n_j} |_U$$

$$\leqq 2C_1 \omega(\varphi_{n_{k+1}}) + 2tC_2 \omega(\varphi_{n_k})/\varphi_{n_k} \leqq (2C_1 + 4C_2)\omega(t).$$

Now, given $\lambda \in \Lambda$, there exists $m_\lambda \in \boldsymbol{N}$ such that $\| T_{n,\lambda} h_n \|_Y \geqq 5C_3/6$ for all $n \geqq m_\lambda$. Therefore (3), (15), (17), (18) deliver

$$\| T_{n_k,\lambda} f_\omega \|_Y \geqq \| T_{n_k,\lambda} \omega(\varphi_{n_k}) h_{n_k} \|_Y - \| T_{n_k,\lambda} g_{k-1} \|_Y - \| T_{n_k,\lambda} \|_{[X,Y]} \| f_\omega - g_k \|_X$$

$$\geqq [5C_3/6 - C_3/3 - 2C_1 M r] \omega(\varphi_{n_k}) \geqq (C_3/6) \omega(\varphi_{n_k})$$

for all $n_k \geqq m_\lambda$. Thus (13) holds true for each $\lambda \in \Lambda$.

Let us point out that the present Theorem may indeed be interpreted as a condensation principle with rates, similar to those given recently in [1] (see also the literature cited there). However, those developed there deal essentially only with countable index sets. On the other hand, the present assumptions upon the elements h_n seem to be rather restrictive in comparison with [1].

Let us conclude with a first application. Choose $X = C[0, 1]$, the space of functions continuous on $[0, 1]$, $Y = \boldsymbol{R}$, the real line, and

$$U := C^{(2)}[0, 1] := \{ f \in C[0, 1];\ f', f'' \in C[0, 1] \},\quad | f |_U := \| f'' \|_C.$$

Then the corresponding K-functional turns out to be equivalent to the usual second modulus of continuity of functions, i. e., if

$$\omega^*(t, f) := \sup_{0 \leqq h \leqq t}\ \sup_{h \leqq x \leqq 1-h} | f(x-h) - 2f(x) + f(x+h) |,$$

there exist constants $b_{1,2} > 0$, independent of f and t, such that

$$\text{(19)} \qquad b_1 \omega^*(t, f) \leqq K(t^2, f;\ C[0, 1],\ C^{(2)}[0, 1]) \leqq b_2 \omega^*(t, f).$$

For the approximate calculation of the indefinite integral $\int_0^x f(u)du$ consider the formula

$$Q_n(f;\ x) := n^{-1}[f(0)/2 + \sum_{j=1}^{k-1} f(j/n) + f(k/n)/2]$$

$$+ f(k/n)(x - k/n) + n[f((k+1)/n) - f(k/n)](x - k/n)^2/2$$

for $x \in [k/n, (k+1)/n]$, $k = 0, 1, \ldots, n-1$, which in case $x = k/n$ coincides with the compound trapezoidal rule. The remainders $(\lambda = x \in [0, 1] = \Lambda)$

$T_{n,x}f := x^{-1}[Q_n(f;\ x) - \int_0^x f(u)du]$ then satisfy (3), (4) with $\varphi_n = 1/n^2$, namely $(M \geqq 2)$

$$|T_{n,x}f| \leqq \begin{cases} 2\|f\|_C & (f \in C[0,\ 1]), \\ (M/n^2)\|f''\|_C & (f \in C^{(2)}[0,\ 1]). \end{cases}$$

Thus in view of the Proposition and (19) one has

$$(20) \qquad |Q_n(f;\ x) - \int_0^x f(u)du| \leqq b_2 M x \omega^*(1/n,\ f)$$

for all $f \in C[0,\ 1]$, $n \in \mathbf{N}$, and $x \in [0,\ 1]$.

Now an application of the condensation principle shows the sharpness of (20). Indeed, consider the functions $h_n(u) = \sin^2(\pi n u)$ which of course satisfy (10) and (11) (with $\varphi_n = 1/n^2$). Since $Q_n(h_n;\ x) = 0$ for all $x \in [0,\ 1]$,

$$|T_{n,x}h_n| = |x^{-1}\int_0^x \sin^2(\pi n u)du| \geqq x^{-1}\int_0^{k/n} \sin^2(\pi n u)du = \frac{k}{\pi n x}\int_0^{\pi} \sin^2 u\, du \geqq \frac{1}{4}(=C_3),$$

if $x \in [k/n,\ (k+1)/n]$, $k \geqq 1$. Hence by the Theorem (with $1/x \leqq m_x \in \mathbf{N}$)

Corollary. *For each modulus ω satisfying (6), (9) there exists $f_\omega \in C[0, 1]$ such that $\omega^*(t,\ f_\omega) = O(\omega(t^2))$ and*

$$(21) \qquad \limsup_{n\to\infty} |Q_n(f_\omega;\ x) - \int_0^x f_\omega(u)du| / \omega(1/n^2) \geqq x/24$$

simultaneously for all $x \in [0,\ 1]$.

Let us mention that an application of those condensation principles with rates given in [1] would only ensure (21) for a dense set of second category in [0, 1].

REFERENCES

1. W. Dickmeis, R. J. Nessel. Condensation principles with rates. *Studia Math.*, 75, 1982, 55-68.

Rhein-Westf. Technische Hochschule
Lehrstuhl A für Mathematik
5100 Aachen BRD

Received on June 15, 1981