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LIDSTONE-TYPE FORMULAS AND NON-HARMONIC SINE EXPANSIONS

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Summary. Some aspects of the eigenvalue problem $y''-\lambda y=0$, $y(0)=0$, $\Phi(y)=0$ with an arbitrary linear functional Φ on $C^1[0, 1]$, connected with the approximation theory, are treated. Generalized Lidstone polynomials are introduced, and explicit representations of the remainder terms of the corresponding Taylor-Lidstone formulas are found. They resemble the integral Cauchy remainder term for the usual Taylor formula, but with a new convolution, found by the author. By means of this convolution a system of projectors onto the eigenspaces of the spectral problem is given explicitly, which is equivalent to a construction of the biorthogonal system of the system of the eigen- and associate functions. In the case of simple eigenvalues the explicit coefficient formulas of the corresponding sine expansion are exhibited. An application to the Samarskii-Ionkin eigenvalue problem is also made.

Let Φ be a linear functional, defined on $C^1[0, 1]$, such that $\lambda=0$ is not an eigenvalue of the spectral problem

(1) $$y''-\lambda y=0, \quad y(0)=0, \quad \Phi(y)=0.$$

This reduces to the restriction $\Phi\{x\}\neq 0$ and without any loss of generality we may assume the normalization condition

(2) $$\Phi\{x\}=1.$$

1. Lidstone-type Formulas.

D e f i n i t i o n 1. The polynomials $\Lambda_n(x)$, $n=0, 1, 2, \ldots$, defined recurrently by $\Lambda''_n(x)=\Lambda_{n-1}(x)$, $\Lambda_0(x)=x$, with the boundary value conditions $\Lambda_n(0)=0$ and $\Phi(\Lambda_n)=0$ are said to be the *generalized Lidstone polynomials*, determined by the functional Φ.

The classical Lidstone polynomials (see e. g. [1]) correspond to the choice $\Phi(f)=f(1)$.

D e f i n i t i o n 2. Let the linear functional Φ satisfy the normalization condition (2). Then the polynomials $\Lambda^*_n(x)$, defined recurrently by $[\Lambda^*_n(x)]''=\Lambda^*_{n-1}(x)$, $\Lambda^*_0(x)=1-\Phi\{1\}x$ with the boundary value conditions $\Lambda^*_n(0)=0$ and $\Phi(\Lambda^*_n)=0$, are said to be the *associated generalized Lidstone polynomials,* determined by the functional Φ.

For the next considerations it is useful to introduce the right inverse operator of d^2/dx^2, corresponding to the problem (1). This is the operator

$$L f(x) = l^2 f(x) - \Phi(l^2 f) x, \tag{3}$$

where for the sake of brevity the denotation

$$l f(x) = \int_0^x f(\xi) d\xi \tag{4}$$

for the integration operator is used. Here we consider L in the space $C[0, 1]$.

The generalized Lidstone polynomials and their associates are the iterates

$$\Lambda_n(x) = L^n\{x\} \quad \text{and} \quad \Lambda_n^*(x) = L^n\{1 - \Phi\{1\} x\}, \tag{5}$$

$n=0, 1, 2, \ldots,$ of L on the functions x and $1-\Phi\{1\}x$, correspondingly.

We aim to develop an algebraic approach to the spectral problem (1) based on a convolution of the operator L, found by Dimovski [2]. Under a convolution of L in $C[0, 1]$ we understand a non-trivial bilinear, commutative, associative operation $f * g$ in $C[0, 1]$, such that L to be a multiplier of the corresponding convolutional algebra, i. e. when the identity

$$L(f * g) = (L f) * g \tag{6}$$

holds in $C[0, 1]$.

Theorem 1. *The operation*

$$(f * g)(x) \tag{7}$$

$$= -\frac{1}{2} \tilde{\Phi}_\xi \{ \int_x^\xi f(\xi + x - \eta) g(\eta) d\eta - \int_{-x}^\xi f(|\xi - x - \eta|) g(|\eta|) \operatorname{sgn}(\xi - x - \eta) \eta . d\eta \}$$

with $\tilde{\Phi} = \Phi \circ l$ *is a convolution of the operator* L*, defined by* (3) *in* $C[0, 1]$, *such that*

$$L f = \{x\} * f. \tag{8}$$

Proof. Here we shall outline a proof, slightly different from those proposed in [2]. Since the bilinearity and commutativity of (7) are evident, only the associativity needs a proof. Let us introduce the entire function of exponential type

$$E(z) = \Phi_\xi \{\sin \xi z\}. \tag{9}$$

If $\lambda \neq \mu$, then by a direct calculation it can easily be shown that

$$\{\sin \lambda x\} * \{\sin \mu x\} = \frac{E(\lambda) \sin \mu x - E(\mu) \sin \lambda x}{\lambda^2 - \mu^2}. \tag{10}$$

Then it is easy to verify the special associativity relation

$$(\{\sin \lambda x\} * \{\sin \mu x\}) * \{\sin \nu x\} = \{\sin \lambda x\} * (\{\sin \mu x\} * \{\sin \nu x\}). \tag{11}$$

Then, differentiating (11) $2m$ times in λ, $2n$ times in μ and $2p$ times in ν, we get an identity which can be written in the form

$$(\{x^{2m}\frac{\sin\lambda x}{\lambda}\}*\{x^{2n}\frac{\sin\mu x}{\mu}\})*\{x^{2p}\frac{\sin\nu x}{\nu}\}$$

$$=\{x^{2m}\frac{\sin\lambda x}{\lambda}\}*(\{x^{2n}\frac{\sin\mu x}{\mu}\}*\{x^{2p}\frac{\sin\nu x}{\nu}\}).$$

Letting $\lambda\to 0$, $\mu\to 0$ and $\nu\to 0$, we get

(12) $$(\{x^{2m+1}\}*\{x^{2n+1}\})*\{x^{2p+1}\}=\{x^{2m+1}\}*(\{x^{2n+1}\}*\{x^{2p+1}\}),$$

valid for $m, n, p=0, 1, 2, \ldots$ Using uniform approximation by odd polynomials, we can now establish the associativity relation $(f*g)*h=f*(g*h)$ for $f, g, h\in C[0, 1]$ with $f(0)=g(0)=h(0)=0$, due to the continuity of (7). In order to accomplish the proof of the associativity of (7) in $C[0, 1]$ we can either refer to the density of the subspace $C_0[0, 1]$ of the functions with $f(0)=0$ in $L^1[0, 1]$, or use a direct approach, proving the special associativity relations $(1*g)*h=1*(g*h)$ and $(1*1)*h=1*(1*h)$ for $g, h\in C_0[0, 1]$, following the above pattern. The details are elaborated in [3]. Identity (8) can be verified by a direct check.

For a special class of linear functionals Φ a convolution of L with the function $\{x\}$ as unit element can be proposed,

Theorem 2. *Let the linear functional* Φ *in* $C^1[0, 1]$, *normed by* $\Phi\{x\}=1$, *be of the form* $\Phi=\Psi\circ l$, *where* Ψ *is a linear functional on* $C[0, 1]$. *Then the operation*

(13) $$(f\tilde{*}g)(x)=f(x)\Phi(g)+g(x)\Phi(f)-\Psi\{1\}\int_0^x f(x-\xi)g(\xi)d\xi$$

$$-\frac{1}{2}\Psi_\xi\{\int_x^\xi f(\xi+x-\eta)g(\eta)d\eta-\int_{-x}^{\xi} f(|\xi-x-\eta|)g(|\eta|)\operatorname{sgn}(\xi-x-\eta)\eta.d\eta\}$$

is a convolution of L *in* $C[0, 1]$, *such that* $\{x\}*f=f$.

Proof. It is not difficult to be shown that under the hypothesis of the theorem the function $f*g$ is from $C^2[0, 1]$, and $f\tilde{*}g=(f*g)''$. The associativity of (13) can be shown in the same manner, as those of (7). The verification of the relation $\{x\}*f=f$ is a matter of a direct check.

Theorem 3. *If* $f(x)\in C^{2n}[0, 1]$, *then the identity*

(14) $$f(x)=\sum_{k=0}^{n-1}[\Phi\{f^{(2k)}\}\Lambda_n(x)+f^{(2k)}(0)\Lambda_n^*(x)]+\Lambda_{n-1}(x)*f^{(2n)}(x)$$

holds.

Proof. If we denote $D=d^2/dx^2$, then let us use the generalized Taylor formula (see [4])

(15) $$I=\sum_{k=0}^{n-1}L^kFD^k+L^nD^n,$$

where

(16) $$Ff(x)=\Phi(f)x+f(0)[1-\Phi\{1\}x]$$

is the initial projector of the right inverse operator L of d^2/dx^2.

Next we transform the remainder term

$$R_n(f)=L^n f^{(2n)}=L^{n-1}(L f^{(2n)})=L^{n-1}(\{x\} * f^{(2n)})=(L^{n-1}\{x\}) * f^{(2n)}$$
$$=\Lambda_{n-1}(x) * f^{(2n)}(x),$$

using representation (5). The theorem is proved.

Using (7), we can write the remainder term of Taylor-Lidstone formula (14) in the following explicit form

$$\text{(17)}\qquad R_n(f)=-\frac{1}{2}\tilde{\Phi}_\xi\{\int_x^\xi \Lambda_{n-1}(\xi+x-\eta)f^{(2n)}(\xi)\,d\xi$$
$$-\int_{-x}^{\xi}\Lambda_{n-1}(\xi-x-\eta)f^{(2n)}(|\eta|)\operatorname{sgn}\eta\,.\,d\eta\},$$

since $\Lambda_{n-1}(x)$ is an odd function.

Theorem 4. *If $\Phi(f)=f(1)$, and $f\in C^{2n-1}[0, 1]$, then the identity*

$$\text{(18)}\qquad f(x)=\sum_{k=1}^{n-1}[f^{(2k)}(1)\Lambda_k(x)+f^{(2k)}(0)\Lambda_k(1-x)]$$
$$+\frac{1}{2}\int_0^1[\bar{\Lambda}_{n-1}(1-x-\xi)+\bar{\Lambda}_{n-1}(1-x+\xi)]f^{(2n-1)}(\xi)d\xi$$

holds with $\bar{\Lambda}_{n-1}(x)$ equal to the odd continuation of $\Lambda_{n-1}(x)$, $0\leq x\leq 1$ with respect to the points $x=1$ and $x=0$ to the segment $[-1, 2]$.

Proof. Let first assume $f\in C^{2n}[0, 1]$. Then by (14) we get

$$R_n(f)=-\frac{1}{2}\int_0^1[\int_x^\xi \Lambda_{n-1}(\xi+x-\eta)f^{(2n)}(\eta)d\eta$$
$$-\int_{-x}^{\xi}\Lambda_{n-1}(\xi-x-\eta)f^{(2n)}(|\eta|)\operatorname{sgn}\eta\,.\,d\eta]\,d\xi,$$

where $\Lambda_{n-1}(x)$ is the classical Lidstone polynomial. By some tedious but elementary transformations we get

$$R_n(f)=\frac{1}{2}\int_x^1[\Lambda_{n-1}(1-x-\xi)-\Lambda_{n-1}(1+x-\xi)]f^{(2n-1)}(\xi)\,d\xi$$
$$+\frac{1}{2}\int_0^x[\Lambda_{n-1}(1-x-\xi)+\Lambda_{n-1}(1-x+\xi)]\,f^{(2n-1)}(\xi)d\xi.$$

It can be written in a shorter way, provided we introduce the odd continuation of the function $\Lambda_{n-1}(x)$, $0\leq x\leq 1$ with respect to the point $x=1$, viz. $\bar{\Lambda}_{n-1}(x)=\Lambda_{n-1}(1-|1-x|)\,.\operatorname{sgn}(1-x)$. Then we get

$$R_n(f)=\frac{1}{2}\int_0^1[\Lambda_{n-1}(1-x-\xi)+\bar{\Lambda}_{n-1}(1-x+\xi)]\,f^{(2n-1)}(\xi)\,d\xi.$$

It remains to note that for the classical Lidstone polynomials $\Lambda_k(x)$ the relation $\Lambda_k^*(x)=\Lambda_k(1-x)$ holds (see [1]), that they are odd functions, and to extend formula (18) from $C^{2n}[0, 1]$ to $C^{2n-1}[0, 1]$. It can be done in an obvious way using the denseness of $C^{2n}[0, 1]$ in $C^{2n-1}[0, 1]$.

Example 1. As an immediate application of formula (14) let us consider the case $\Phi(f)=f'(1)$. The corresponding Lidstone-type polynomials are studied by Pethe and Sharma [4]. The Lidstone-type formula

$$f(x)=\sum_{k=0}^{n-1}[f^{(2k+1)}(1)\Lambda_k(x)+f^{(2k)}(0)\Lambda_k^*(x)]+R_n(f)$$

with

$$R_n(f)=-\frac{1}{2}\int_x^1\Lambda_{n-1}(1+x-\xi)f^{(2n)}(\xi)\,d\xi+\frac{1}{2}\int_{-x}^1\Lambda_{n-1}(1-x-\xi)f^{(2n)}(|\xi|)\,\mathrm{sgn}\,\xi\cdot d\xi$$

holds true, when $f\in C^{2n}[0, 1]$.

Then generalized Lidstone polynomials, introduced by Definition 1, have the following generating function

$$\sum_{n=0}^{\infty}\Lambda_n(x)z^{2n}=i\,\mathrm{sh}\,zx/E(iz)\quad\text{with}\quad E(z)=\Phi_\xi\{\sin\xi z\}. \tag{19}$$

For the associated generalized Lidstone polynomials we have

$$\sum_{n=0}^{\infty}\Lambda_n^*(x)z^{2n}=i\Phi_\xi\{\mathrm{sh}\,z(\xi-x)\}/E(iz). \tag{20}$$

2. Eigenexpansions for the Spectral Problem (1). Now, let us consider the spectral problem (1) which well can be a non-local one. In a sense, the convolutions (7) and (13) can be considered as "solutions" of this problem. Our aim is to make clear this somewhat vague statement. We assume that the entire function of exponential type

$$E(z)=\Phi_\xi\{\sin\xi z\} \tag{21}$$

has an infinite sequence $z_1, z_2, z_3, \ldots, z_n, \ldots$ of different zeros $z_n\neq 0$. Since $E(z)$ is an odd function, it has along with z_n also $-z_n$ as a zero. We assume that in the above sequence of each such pair z_n, $-z_n$ only one of the zeros is taken. This is justified by the fact that z_n and $-z_n$ give the same eigenvalue $\lambda_n=-z_n^2$. The multiplicity of z_n as a zero of $E(z)$ is said to be the multiplicity of the corresponding eigenvalue $\lambda_n=-z_n^2$. Let $\varkappa_n$, $n=1, 2, \ldots$, be these multiplicities. If $\varkappa_n>1$, then the eigenspace

$$K_n=\ker(I-\lambda_n L)^{\varkappa_n} \tag{22}$$

corresponding to this eigenvalue contains not only the multiples of $\sin z_n x$' but $\varkappa_n-1$ more associated eigenfunctions, and K_n in all cases is $\varkappa_n$-dimensional.

The problem of expanding the functions from $C[0, 1]$ into series on the eigen- and associated eigenfunctions of the spectral problem (1) reduces to the problem of constructing of appropriate spectral projectors of $C[0, 1]$ onto each of the eigenspaces K_n. The following theorem gives these projectors in explicit form as convolution operators with respect to convolution (7).

T h e o r e m 5. *Let* z_n *be a zero of* $E(z)$ *and let* C_n *be a simple contour around* z_n, *which does not contain any other zero of* $E(z)$. *Then the convolutional operator*

$$P_n f = \varphi_n * f \quad with \quad \varphi_n(x) = \frac{1}{\pi i} \int_{C_n} \frac{\lambda \sin \lambda x}{E(\lambda)} d\lambda \tag{23}$$

is a projector of $C[0, 1]$ *onto the eigenspace* $K_n = \ker (I + z_n^2 L)^{\varkappa_n}$ *of spectral problem* (1), *commuting with* L.

P r o o f. The commutating of P_n with L follows from Theorem 1. Since $L\{\sin \lambda x\} = -\frac{\sin \lambda x}{\lambda^2} + \frac{E(\lambda) x}{\lambda^2}$, then

$$(I + z_n^2 L)^{\varkappa_n} \varphi_n(x) = \frac{1}{\pi i} \int_{C_n} \frac{(1 - z_n^2/\lambda^2)^{\varkappa_n} \lambda \sin \lambda x}{E(\lambda)} d\lambda = 0,$$

since the integrand does not contain poles in C_n. If $f \in C[0, 1]$ is arbitrarily chosen, then $P_n f \in K_n$, since

$$(I + z_n^2 L)^{\varkappa_n} P_n f = [(I + z_n^2 L)^{\varkappa_n} \varphi_n] * f = \{0\} * f = 0$$

due to the fact that the operation $*$ is a convolution of L. Hence P_n maps $C[0, 1]$ onto K_n. It remains to show that P_n is a projector. The projector property $P_n^2 = P_n$ is equivalent to $\varphi_n * \varphi_n = \varphi_n$. In order to show the last relation, we take two simple contours C_n' and C_n'' around z_n, such that C_n' contains C_n'' in its inside. Then

$$\varphi_n * \varphi_n = \frac{1}{(\pi i)^2} \int_{C_n'} \int_{\times C_n''} \frac{\lambda \mu \{\sin \lambda x\} * \{\sin \mu x\}}{E(\lambda) E(\mu)} d\lambda \, d\mu.$$

Now, using (10), we get

$$\varphi_n * \varphi_n = \frac{1}{(\pi i)^2} \int_{C_n''} \frac{\mu \sin \mu x}{E(\mu)} d\mu \int_{C_n'} \frac{\lambda \, d\lambda}{\lambda^2 - \mu^2} - \frac{1}{(\pi i)^2} \int_{C_n'} \frac{\lambda \sin \lambda x}{E(\lambda)} d\lambda \int_{C_n''} \frac{\mu \, d\mu}{\lambda^2 - \mu^2}.$$

Since

$$\int_{C_n'} \frac{\lambda d\lambda}{\lambda^2 - \mu^2} = \pi i \quad \text{and} \quad \int_{C_n''} \frac{\mu d\mu}{\lambda^2 - \mu^2} = 0,$$

then

$$\varphi_n * \varphi_n = \frac{1}{\pi i} \int_{C_n''} \frac{\mu \sin \mu x}{E(\mu)} d\mu = \varphi_n.$$

It remains to show that each element of K_n is invariant under P_n. This follows from the invariance under P_n of the basis

$$\varphi_n, \ (I + z_n^2 L) \varphi_n, \quad (I + z_n^2 L)^2 \varphi_n, \ \ldots, \ (I + z_n^2 L)^{\varkappa_n - 1} \varphi_n$$

of the $\varkappa_n$-dimensional linear space K_n.

Corollary. *If z_n is a simple zero of $E(z)$, then $\varphi_n(x)=\frac{2z_n \sin z_n x}{E'(z_n)}$, and*

(24) $$P_n f = f * \varphi_n = [-\frac{1}{z_n E'(z_n)}\Phi_\xi\{\int_0^\xi \sin z_n(\xi-\eta) f(\eta) d\eta\}] \sin z_n x.$$

In particular,

(25) $$P_n(\Lambda_k(x)) = L^k P_n\{x\} = L^{k+1}\varphi_n = \frac{(-1)^{k+1} 2 \sin z_n x}{z_n^{2k+1} E'(z_n)}.$$

The explicit representation (24) follows easily from (23), using the residue method.

Let $z_1, z_2, \ldots, z_n, \ldots$ be a sequence of zeros of $E(z)$, in which each pair of opposite zeros of $E(z)$ appears once. Then we give the following definition.

Definition 3. Let $f \in C[0, 1]$. The formal expansion

(26) $$f \sim \sum_{n=1}^{\infty} f * \varphi_n$$

is said to be the *eigenfunction expansion of f*, corresponding to eigenvalue problem (1).

If $E(z)$ has only simple zeros, then (26) reduces to

(27) $$f(x) \sim \sum_{n=1}^{\infty} [-\frac{1}{z_n E'(z_n)}\Phi_\xi\{\int_0^\xi \sin z_n(\xi-\eta) f(\eta) d\eta\}] \sin z_n x$$

and the linear functionals

(28) $$\chi_n(f) = -\frac{1}{z_n E'(z_n)}\Phi_\xi\{\int_0^\xi \sin z_n(\xi-\eta) f(\eta) d\eta\}, \quad n=1, 2, 3, \ldots,$$

form the biorthogonal system of the sine system $\{\sin \lambda_n x\}_{n=1}^{\infty}$.

The projector system (23) contains in a compact form the biorthogonal system of the system of eigen- and associated eigenfunctions of spectral problem (1) in the general case, too. For the simpler spectral problem

$$y' - \lambda y = 0, \quad \Phi(y) = 0$$

the corresponding biorthogonal system had been constructed by Delsarte [6] and Leontiev [7].

The first problem, connected with the projector system, is the question whether a function $f \in C[0, 1]$ can be restored from its projections $P_n f = \varphi_n * f$, $n=1, 2, \ldots$, uniquely. To this end some assumptions on the functional Φ in (1) or, equivalently, on the distributions of the zeros of $E(z)=\Phi_\xi\{\sin \xi z\}$ should be made. Here we restrict our considerations on a special but typical example.

3. Samarskii-Ionkin Eigenvalue Problem [8]. This is Problem (1) with

(29) $$\Phi(f) = 2\int_0^1 f(\xi) d\xi.$$

Here $E(z)=\frac{2(1-\cos z)}{z}$ with zeros $z_n=2\pi n$ with multiplicities $\varkappa=2$, $n=1, 2, \ldots$ The eigenspaces $K_n=\ker(I+z_n^2L)^2$ are two-dimensional. Here a better approach is by convolution (13), which here takes the form

$$(30)\quad (f\tilde{*}g)(x)=2\left[\int_0^1 f(\xi)d\xi\right]g(x)+2\left[\int_0^1 g(\xi)\,d\xi\right]f(x)-2\int_0^x f(x-\xi)\,g(\xi)d\xi$$
$$-\int_x^1 f(1+x-\xi)\,g(\xi)d\xi+\int_{-x}^1 f(|1-x-\xi|)\,g(|\xi|)\,\operatorname{sgn}(1-x-\xi)\,\xi d\xi\,.$$

An advantage of this convolution is the relation $\{x\}\tilde{*}f=f$, which expresses the fact that the function $\{x\}$ is an unit of the corresponding convolutional algebra. Now projectors (23) take the form $P_nf=\tilde{\varphi}_n\tilde{*}f$ with $\tilde{\varphi}_n(x)=-2x\cos 2\pi nx$. Thus we get

$$(31)\qquad P_nf(x)=\{-2x\cos 2\pi nx\}*f(x)$$
$$=\left[4\int_0^1 f(\xi)(1-\xi)\sin 2\pi n\xi\,d\xi\right]\sin 2\pi nx-\left[4\int_0^1 f(\xi)(1-\cos 2\pi n\xi)d\xi\right]x\cos 2\pi nx.$$

L e m m a (uniqueness property of projector system (31)). *Let $f\in C[0, 1]$. If $P_nf=0$ for $n=1, 2, \ldots,$ then $f=0$.*

P r o o f. Since $f=0\Leftrightarrow Lf=0$, we may assume $f(0)=0$ and $\int_0^1 f(\xi)d\xi=0$. From $P_nf=0$ we get

$$\int_0^1 f(\xi)(1-\cos 2\pi n\xi)d\xi=0\quad\text{and}\quad\int_0^1 f(\xi)(1-\xi)\sin 2\pi n\xi\,d\xi=0.$$

From the first relation it follows that $f(x)$ is an odd function with respect to the point $x=1/2$, i. e. the relation $f(1-x)=-f(x)$ holds. Using this relation, from the second equality we get $\int_0^1 \xi f(\xi)\sin 2\pi n\xi d\xi=0$ and hence $\int_0^1 f(\xi)\sin 2\pi n\xi d\xi=0$. Therefore $f(x)$ should be an even function with respec to the point $x=1/2$ too, i. e. $f(1-x)=f(x)$. Hence $f(x)\equiv 0$, q. e. d.

Following Definition 3, we can introduce the eigenexpansion

$$(32)\qquad f(x)\sim\sum_{n=1}^{\infty}\left[4\int_0^1 f(\xi)(1-\xi)\sin 2\pi n\xi d\xi\right]\sin 2\pi nx$$
$$-\left[4\int_0^1 f(\xi)(1-\cos 2\pi n\xi)d\xi\right]x\cos 2\pi nx.$$

This expansion may well be non-convergent for a function $f(x)$ from $C[0, 1]$, but nevertheless it determines uniquely the corresponding function. Here we state a simple sufficient condition for convergence of the series in (32) to the function $f(x)$.

T h e o r e m 6. *Let $f\in C^2[0, 1]$ be such that $f(0)=0$, and $\int_0^1 f(\xi)d\xi=0$. Then*

$$(33)\qquad f(x)=\sum_{n=1}^{\infty}\left[4\int_0^1 f(\xi)(1-\xi)\sin 2\pi n\xi d\xi\right]\sin 2\pi nx$$
$$-\left[4\int_0^1 f(\xi)(1-\cos 2\pi n\xi)\,d\xi\right]x\cos 2\pi nx,$$

the series being uniformly convergent on [0, 1].

P r o o f. Under the hypothesis, $f(x)=Lf''(x)$, and using (30), we can write $f(x)=(L\{x\})*f''(x)=\Lambda_1(x)*f''(x)$. By a direct check it is seen that

$$(34) \qquad \Lambda_1(x)=\frac{x^3}{6}-\frac{x}{12}=\sum_{n=1}^{\infty}\left\{\frac{2}{(2\pi n)^2}x\cos 2\pi nx-\frac{4}{(2\pi n)^3}\sin 2\pi nx\right\}$$

with the eigenexpansion (32) of $\Lambda_1(x)$ on the right-hand side. The series is uniformly convergent on [0, 1]. The series received by term-by-term convolutional multiplication of (34) by $f''(x)$ is uniformly converegent, too. But it is exactly the Samarskii-Ionkin eigenexpansion of $f(x)$, thus proving the theorem.

N o t e. Most of the results in this paper can be stated for the space $L^1[0, 1]$, too.

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