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SATURATION OF APPROXIMATION OF n-DIMENSIONAL FUNCTIONS

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Summary. The "parabola technique" of Bajšanski and Bojanić was used to derive a small o saturation result for a variety of linear positive approximation processes on $C(I)$, $I\subset R$. We will obtain a similar result for approximation on $C(I)$, $I\subset R^n$. The major difference between the n-dimensional and one-dimensional case is that the trivial class here consists of all solutions of certain elliptic differential equations rather than the linear functions.

1. We will discuss in this paper a sequence A_k of linear positive operators on B. C. (I) (bounded continuous functions on I) to B. C. (I), where $I\subset R^n$, or with some additional restriction, on $C_W(I)$ (locally continuous functions bounded by $KW(x)$ on I) to B. C. (B), where $B\subset I$.

The class of operators satisfies for $x=(x_1, \dots, x_n)$

$$A_k[\varphi_i;\ x]-\varphi_i(x)=o(\sigma_k^2),\quad k\to\infty, \tag{1.1}$$

for $\varphi_0(x)=1$, $\varphi_i(x)=x_i$, $1\leqq i\leqq n$, and

$$A_k[\psi_{ij};\ x]-\psi_{ij}(x)=\theta_{ij}(x)\sigma_k^2+o(\sigma_k^2),\quad k\to\infty, \tag{1.2}$$

for $\psi_{ij}(x)=x_ix_j$, $1\leqq i, j\leqq n$, where $\sigma_k^2=o(1)$, $\theta_{ij}(x)$ are continuous in x in question and

$$\sum_{i,j}\theta_{ij}(x)h_ih_j>0 \text{ for } h_1^2+\dots+h_n^2=1. \tag{1.3}$$

Simple computation will show that (1.1), (1.2) and the positivity of A_k will imply that $\Sigma\theta_{ij}(x)h_ih_j\geqq 0$. The strict inequality (1.3) is however an additional assumption that will be required in most cases on I^0. For this class of operators with minor restriction we shall prove $A_k[f, x]-f(x)=o_x(\sigma_k^2)$ on some ball in I^0, in which A_k satisfies (1.1), (1.2) and (1.3) if and only if inside that ball $P(D)f(x)=\Sigma\theta_{ij}(x)\partial^2 f/\partial x_i\partial x_j=0$. The set of such functions on B is not dense in $C(B)$. Some examples and applications will be given.

2. The Parabola Technique Generalized. To achieve the local saturation result we shall generalize here the parabola technique.

Theorem 2.1. *Let B be a closed ball in R^n, let $f\in C(B)$, $f(x)=0$ on ∂B and $f(y_0)>0$ for some $y_0\in B$, then there exists a function*

$$Q(x)=\alpha(\sum_{i=1}^{n} x_i^2)+\sum_{i=1}^{n}\beta_i x_i+\gamma \text{ such that } \alpha<0,\ Q(x)>0 \text{ on } B,$$

$Q(x)\geq f(x)$ on B and $Q(y)=f(y)$ for some y in B^0 (interior of B).

Proof. Let M be the maximum of $f(x)$ in B, $M>0$. Let $Q_1(x)$ be given by $\alpha\Sigma(x_i-x_{ic})^2+M-\alpha r^2=Q_1(x)$, where r is the radius of B, x_{ic} are coordinates of the center of B, and $\alpha<0$ so small that $-\alpha r^2<f(y_0)$. We have $Q_1(x)\geq M$ and $\min_{x\in B}[Q_1(x)-f(x)]=m$. Define $Q(x)=Q_1(x)-m$. Obviously $Q(x)\geq f(x)$ (by definition of m) and $Q(x)$ is of the right form. It is obvious that $Q_1(x)$ and therefore $Q_1(x)-f(x)=M$ on ∂B, and moreover that $Q_1(y_0)-f(y_0)\leq M-\alpha r^2-f(y_0)<M$ and therefore m, the minimum of $Q_1(x)-f(x)$, is achieved on B^0. On ∂B now $Q(x)=M-m>0$. We choose y to be such that $Q_1(y)-f(y)=m$.

3. A Voronovskaya-type Relation. The parabola technique was introduced to prove a small o saturation theorem for Bernstein polynomials by Bajšanski and Bojanic [1] and more general one-dimensional cases were treated (see [2]) later. However, for the n-dimensional case we shall need that $A_k(\psi;x)-\psi(x)=o(\sigma_k^2)$, at least locally, for ψ satisfying the elliptic equation $\Sigma\theta_{ij}(x)\partial^2\psi/\partial x_i\partial x_j=0$. The analogous condition for R_1 is that for a linear function ψ $A_k(\psi;x)-\psi(x)=o(\sigma_k^2)$, since in one dimension $\theta(x)\psi''(x)=0$ means that ψ is linear. We shall prove that under certain conditions the Voronovskaya relation

$$\sigma_k^{-2}[A_k(f;x)-f(x)]=\Sigma\theta_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i\partial x_j}+o(1)$$

is valid, which implies our requirement. (One should note that in the one-dimensional problem $A_k(\psi;x)-\psi(x)=o(\sigma_k^2)$ is assumed for any ψ that belongs to the trivial class.)

Theorem 3.1. *Let $A_k(f;x)$ be a sequence of positive linear operators B. C. (I) satisfying (1.1) and (1.2) for some $x\in B^0$, $B\subset I$, and let*

(3.1) $$A_k(|t_i-x_i|^\alpha;x)=o(\sigma_k^2)$$

for $1\leq i\leq n$ and some $\alpha>2$ for some $x\in B^0$, then for $f\in C^2(B)$

(3.2) $$\sigma_k^{-2}[A_k(f;x)-f(x)]=P(D)f(x)+o(1),\quad k\to\infty,$$

where

(3.3) $$P(D)f=\sum_{i,j}\theta_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i\partial x_j}.$$

If in addition in (1.1), (1.2) and (3.1) the small o is uniform in $B_1\subset B^0$, then it is in (3.2).

Remark 3.1. In most applications the interesting α in (3.1) is 4.

Remark 3.2. We actually have an estimate here for all functions bounded by $K(\Sigma|x_i|^\alpha)$ not just bounded.

Remark 3.3. We did not require (1.1), (1.2) and (3.1) to be valid outside of B.

Proof. For a point $a=(a_1,\ldots,a_n)\in B^0$, for which (1.1), (1.2) and (3.1) are satisfied, we denote

(3.4) $$I(f,x,a)=\sum_{i=1}^{n}(x_i-a_i)\frac{\partial f(a)}{\partial x_i}+\frac{1}{2}\sum_{i,j}(x_i-a_i)(x_j-a_j)\frac{\partial^2 f(a)}{\partial x_i\partial x_j}.$$

Using the Taylor formula, we observe for $|x-a|<\delta$ (uniformly for $a\in B_1$) $-\varepsilon\Sigma_{i,j}|x_i-a_i||x_j-a_j|<f(x)-f(a)-I(f,x,a)<\varepsilon\Sigma_{i,j}|x_i-a_i||x_j-a_j|$. We recall that $\Sigma_{i,j}|x_i-a_i||x_j-a_j|\leq n\Sigma_{i=1}^{n}(x_i-a_i)^2$ and that $\Sigma|x_i-a_i|^2\leq(\Sigma|x_i-a_i|^{\alpha})^{2/\alpha}n^{\alpha-2/\alpha}$, or $\Sigma|x_i-a_i|^{\alpha}\geq n^{2-\alpha/2}(\Sigma|x_i-a_i|^2)^{\alpha/2}$. We also have $|f(x)-f(a)|<M_1$ (f is bounded)

$$\max_i\left|\frac{\partial f(a)}{\partial x_i}\right|\leq M_2 \quad\text{and}\quad \max_{i,j}\left|\frac{\partial^2 f(a)}{\partial x_i\partial x_j}\right|\leq M_3$$

(and the last two are bounded uniformly for $a\in B_1$). Therefore for $|x-a|\geq\delta$

$$|f(x)-f(a)|<\frac{M_1}{\delta^{\alpha}}(\Sigma|x_i-a_i|^2)^{\alpha/2}\leq\frac{M_1}{\delta^{\alpha}n^{2-\alpha/2}}\Sigma|x_i-a_i|^{\alpha}$$

and

$$|I(f,x,a)|<M_2\Sigma|x_i-a_i|+M_3\Sigma|x_i-a_i||x_j-a_j|\leq M_2n^{1/2}(\Sigma|x_i-a_i|^2)^{1/2}$$

$$+M_3n\Sigma|x_i-a_i|^2\leq\left(\frac{M_2n^{1/2}}{\delta^{\alpha-1}}+\frac{M_3n}{\delta^{\alpha-2}}\right)(\Sigma|x_i-a_i|^2)^{\alpha/2}\leq\left(\frac{M_2n^{1/2}}{\delta^{\alpha-1}}+\frac{M_3n}{\delta^{\alpha-2}}\right)n^{(\alpha-2)/2}\Sigma|x_i-a_i|^{\alpha}.$$

Therefore

$$-\varepsilon n\Sigma(x_i-a_i)^2-K\Sigma|x_i-a_i|^{\alpha}<f(x)-f(x_0)-I(f,x,x_0)<\varepsilon n\Sigma(x_i-a_i)^2+K\Sigma|x_i-a_i|^{\alpha}.$$

We complete the proof, when we multiply by σ_k^{-2}, recall that A_k is positive and use (1.1), (1.2) and (3.1).

We can discuss without much additional difficulty the Voronovskaya type relation for functions of some growth rather than bounded.

Theorem 3.2. *Let $A_k(f;x)$ be a sequence of positive linear operators on locally continuous functions bounded by $(1+|x|^{\alpha})w(x)$ for some $w(x)\geq 1$ and $\alpha>2$, the relations*

(3.5) $$A_k(|t_i-x_i|^{\alpha}w(t);\,x)=o(\sigma_k^2)\quad k\to\infty \text{ for all } i,$$

(1.1) *and* (1.2) *be valid for some $x\in B^0$ and $f\in C^2(B)$, then* (3.2) *is valid. The Voronovskaya* (3.2) *holds uniformly in $B_1\subset B^0$, if the same is true for* (1.1), (1.2) *and* (3.5).

Remark 3.4. We assume $|f(x)|\leq K(1+|x|^{\alpha})w(x)$, since by (3.5) it is clear that on functions of this type A_k is defined.

Proof. We have to change only the estimate for $|x|>\delta$ and there only for $|f(x)-f(a)|$. We have $|f(x)-f(a)|\leq K(1+|x|^{\alpha})w(x)\leq K_1(\delta)\Sigma|x_i-a_i|^{\alpha}w(x)$.

4. The Saturation Result. The main result of our paper follows now.

Theorem 4.1. *Let A_k be a sequence of positive linear operators on B. C.(I) to $C(B)$, B a ball $B\subset I\subset R^n$ and let* (1.1), (1.2) *and* (3.1) *be satisfied for $x\in B$, let $\theta_{ij}(x)$ be continuous on B and $\Sigma\theta_{ij}(x)h_ih_j>0$ on B for $|h|=1$, then*

(4.1) $$A_k[f;x]-f(x)=o_x(\sigma_k^2) \text{ for all } x\in B^0 \text{ if and only if}$$

$$P(D)f(x)=0 \text{ for all } x\in B^0.$$

We can state the theorem for operations on functions with certain growth. We first define the space $C_{\alpha, w}(I)$

$$C_{\alpha, w}(I)=\{f;\ f \text{ continuous and } |f(x)|\leq K(1+|x|^{\alpha})w(x)\}.$$

Theorem 4.2. *Let A_k be a sequence of positive linear operators from $C_{\alpha, w}(I)$ to functions on $B\subset I$ satisfying on B* (1.1), (1.2) *and* (3.5) *and let $\theta_{ij}(x)$ be continuous on B satisfying* (1.3) *there, then* (4.1) *follows.*

Proof. The "if" part of Theorem 4.1 and 4.2 follows the corresponding Voronovskaya relation, that is Theorems 3.1 and 3.2 respectively.

For the "only if" part, we observe that with no loss of generality we can show the result in B_1, a ball concentric with B and strictly inside B. On ∂B_1 $f(x)$ is continuous and therefore there is a solution of $\Sigma\theta_{ij}(x)(\partial^2 g/\partial x_i\partial x_j)=0$ in B_1 such that $f-g=0$ on ∂B_1. In B_1 either $f=g$ in which case we completed the proof or for $F=f-g$ there exists a maximum or minimum in B_1^0 say maximum. We may define g outside B_1 to be continuous and bounded (for instance set $g(x)=0$ on ∂B and B^c and linearly on each radius between ∂B_1 and ∂B). Therefore by the Voronovskaya result (Theorems 3.1 and 3.2), $A_k(F, x)-F(x)=o_x(\sigma_k^2)$ in B_1^0. Using Theorem 2.1, we observe that there exists $Q(x)$, $Q(x)=\alpha(\Sigma_{i=1}^n x_i^2)+\Sigma_{i=1}^n\beta_i x_i+\gamma$ with $\alpha<0$ such that $Q(x)>F(x)$ on B_1 and for some y $y\in B_1^0$ $Q(y)=F(y)$.

$$A_k(F, y)-F(y)=A_k(F, y)-Q(y)\leq A_k(Q(x)+\max(F(x)-Q(x), 0);\ y)-Q(y)$$

$$\leq\{A_k(Q;\ y)-Q(y)\}+A_k(\max(F(x)-Q(x), 0);\ y)=I_1+I_2.$$

It is clear that $I_1=A_k[Q, y]-Q(y)=\alpha\Sigma\theta_{ii}(y)\,.\,\sigma_k^2+o(\sigma_k^2)$, where α is negative and $\theta_{ii}(y)>0$ for all i because of the ellipticity. Therefore, in order to derive a contradiction, it would suffice to show that $I_2=o(\sigma_k^2)$. Following the technique of the preceding section, we observe that for any $K>0$ $K\Sigma|x_i-y_i|^{\alpha}w(x)\geq K\Sigma|x_i-y_i|^{\alpha}\geq\max(F(x)-Q(x), 0)$ for $|x-y|\leq\delta$ and some $\delta>0$, since for δ sufficiently small the right-hand side is 0. For $|x-y|>\delta$ $\max(F(x)-Q(x), 0)\leq|F(x)|+|Q(x)|$ and we can choose K big enough so that $K\Sigma|x_i-y_i|^{\alpha}\geq|F(x)|+|Q(x)|$ for $|x-y|\geq\delta$ in case we are proving Theorem 4.1 and K big enough so that $K\Sigma|x_i-y_i|^{\alpha}w(x)\geq|F(x)|+|Q(x)|$ when proving Theorem 4.2. We now use (3.1) and (3.5) to obtain $I_2=o(\sigma_k^2)$ for Theorems 4.1 and 4.2 respectively.

5. Applications. A. In [4] a class of approximation processes, satisfying

$$A_k(f;\ x)=\int_{R^n} f(x-t)d\mu_k(t), \tag{5.1}$$

where $d\mu_k(t)$ is positive, $t=(t_1, \ldots, t_n)$, $x=(x_1, \ldots, t_n)$

$$\int_{R^n} d\mu_k(t)=1, \quad \sigma_k^{-2}\int_{R^n} t_i d\mu_k(t)=o(1), \quad k\to\infty \tag{5.2}$$

and

$$\sigma_k^{-2}\int_{R^n} t_i t_j d\mu_k(t)=A_{ij}+o(1), \quad k\to\infty \text{ where } \det A_{ij}\neq 0. \tag{5.3}$$

If we assume also for some $\alpha>2$

$$\sigma_k^{-2}\int_{R^n}|t_i|^{\alpha}d\mu_k(t)=o(1), \quad k\to\infty, \tag{5.4}$$

we obtain $A_k(f, x)-f(x)=o_x(\sigma_k^2)$ for $x\in B$ if and only if $\Sigma A_{ij}\dfrac{\partial^2 f(x)}{\partial x_i\partial x_j}=0$ there.

This is more general than [4] in a way, since $A_k(f;\, x)-f(x)$ does not have to be estimated uniformly in order to obtain the result. Moreover, if we take for instance the Gauss-Weierstrass approximation

$$G_k * f=\gamma_k \int_{R^n} \exp(-k(x-y)^2) f(y)dy, \tag{5.5}$$

we can use Theorem 4.2 with $w(x)=e^{M|x|^2}$ and $\alpha=4$ and if $G_k * f-f=o_x(k^{-1})$ for all x, then we have $\Delta f=0$ (of course f also satisfies $|f(x)|\leqq(1+x^4)Ke^{Mx^2}$). This is not at odds with the well-known result $\|G_k * f-f\|_C=o(k^{-1})$ implies $f(x)=c$, since, if $f\in$ B. C. (R^n) and $\Delta f=0$ everywhere, then $f(x)=c$.

B. The Bernstein multidimensional polynomial approximation is given by

$$B_k(f, x) = \sum_{\frac{\nu}{k}\in\Delta_n} f\left(\frac{\nu}{k}\right)\binom{k}{\nu}\varphi_\nu(x), \tag{5.6}$$

where $\nu=(\nu_1, \ldots, \nu_n)$, $\binom{k}{\nu}=k!/\nu_1!\ldots\nu_n!\,(k-\nu_1-\cdots-\nu_n)!$,

$\varphi_\nu(x)=x_1^{\nu_1}\ldots x_n^{\nu_n}(1-x_1\ldots x_k)^{k_i-\nu_1-\cdots-\nu_k}$ and $\Delta_n=\{x;\ x_i\geqq 0$ and $\Sigma x_i\leqq 1\}$.

Miccheli [6] found $\|B_k(f, x)-f(x)\|_{\Delta_n}=o(k^{-1})$ implies f is linear. For the local result we have in this case $B_k(f, x)-f(x)=o_x(k^{-1})$ for all x in some interior ball to Δ_n implies $P(D)f(x)\equiv \sum_{i=1}^{n} x_i(1-x_i)\frac{\partial^2 f}{\partial x_i^2}-\Sigma_{i\neq j}x_i x_j\frac{\partial^2 f}{\partial x_i \partial x_j}=0$ there. The equation $P(D)f$ is elliptic in Δ_n^0, since

$$\sum_{i=1}^{n} x_i(1-x_i)h_i^2-2\sum_{i<j}x_i x_j h_i h_j\geqq \sum_{i=1}^{n} x_i(1-x_i)h_i^2-\sum_{i<j}x_i x_j(h_i^2+h_j^2)$$

$$=\sum_{i=1}^{n} x_i(1-\Sigma_{j\neq i}x_i)h_i^2,$$

and for $x\in\Delta_n^0$ the coefficients of h_i^2 are always strictly positive. It is simple but tedious calculation to verify (1.1) and (1.2). One should also note that because of the singularity on $\partial\Delta_n$ the only continuous functions that satisfy $P(D)f=0$ everywhere are the linear functions. Locally, however, the situation is different.

C. Szasz and Baskakov. One can generalize Szasz and Baskakov operators to

$$S_k(f, x)=\sum_{\nu_1=0}^{\infty}\cdots\sum_{\nu_n=0}^{\infty} f\left(\frac{\nu_1}{k},\ldots,\frac{\nu_n}{k}\right)e^{-k(x_1+\cdots+x_n)}\frac{(kx_1)^k}{k!}\cdots\frac{(kx_n)^k}{k!} \tag{5.7}$$

and

$$V_k(f, x)=\sum_{\nu_1=0}^{\infty}\cdots\sum_{\nu_k=0}^{\infty} f\left(\frac{\nu_1}{k},\ldots,\frac{\nu_k}{k}\right)\prod_{i=1}^{n}\binom{k+\nu_i-1}{\nu_i}x_i^{\nu_i}(1+x)^{-k-\nu_i}. \tag{5.8}$$

Simple but tedious calculation will yield for bounded $f(x)$, $I=\{x_i,\ x_i\geqq 0\}$,

$S_k(f, x)-f(x)=o_x(k^{-1})$ in B if and only if $P_1(D)f=0$ there,

$$V_k(f; x)-f(x)=o_x(k^{-1}) \text{ in } B \text{ if and only if } P_2(D)f=0 \text{ there,}$$

where $P_1(D)f(x)=\sum_{i=1}^{n} x_i \frac{\partial^2 f(x)}{\partial x_i^2}$ and $P_2(D)f(x)=\sum_{i=1}^{n} x_i(1+x_i)\frac{\partial^2 f(x)}{\partial x_i^2}$.

These equations are obviously elliptic in I^0. Calculation following [5] can be used to determine that for $w(x)=e^{A|x|}$ and $\alpha=4$, that is for functions $|f(x)|\leq K(1+|x|^4)e^{A|x|}$

$$S_k(f; x)-f(x)=o_x(k^{-1}) \text{ or } V_k(f; x)-f(x)=o_x(k^{-1}) \text{ for all } x\in B$$

would imply $P_1(D)f=0$ or $P_2(D)f=0$, where $P_1(D)$ and $P_2(D)$ are those given earlier.

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