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ABOUT SOME SOBOLEV SPACES WITH WEIGHT AND THEIR DUALS

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0. Introduction. In the study of gas exhalations there appears (after some reductions) the following parabolic equation:

$$P(x)\frac{\partial u(x,\ t)}{\partial t}=\frac{\partial}{\partial x}\Big(q(x)\frac{\partial u(x,\ t)}{\partial x}\Big),\quad x\in(0,\ 1),\ t>0 \tag{0.1}$$

with a boundary condition

$$\frac{\partial u}{\partial x}(0,\ t)=0,\ u(1,\ t)=0,\quad t\geqq 0, \tag{0.2}$$

and with an initial condition

$$u(x,\ 0)=\delta(x-x_0),\quad x\in\langle 0,\ 1\rangle\quad (x_0\in(0,\ 1)).$$

Here δ is the Dirac distribution (which describes pointwise source of exhalations — "chimney"), and P, q are smooth functions on $\langle 0,\ 1\rangle$, positive for $x>0$, which vanish in the origin (physically P is the wind velocity and q is coefficient of diffusion) [1]. It is possible to establish existence and properties of solution to (0.1) — (0.3) by technique of semigroups considering this problem as an abstract differential equation in a convenient Hilbert space [2]. The space X must describe degeneration of coefficients P, q and simultaneously contain in some sense the Dirac δ-function, which forms an initial condition. To this scope it is convenient to take as X the dual space to certain Sobolev-type space with weight. Namely, the space W is introduced as the space of functions u, which vanish at the point 1 and such that the (distributional) derivative of u/P is a square integrable with the weight q; the space X is the dual to W. The study of the properties of these spaces is the aim of the present paper. These properties differ essentially in dependence on the rate of convergence of P, q at the origin. Namely, if the function q^{-1} is not integrable, then the space W is a normal space of distribution and hence it is possible to consider X as subspace of $\mathscr{D}'(0,\ 1)$, which contains δ-function. If it is not the case, then the space $\mathring{W}=\overline{\mathscr{D}(0,\ 1)}$ forms a proper subset of W (which can easily be characterized with the help of a "trace theorem") and the structure of X is more complicated. In the original physical statement both cases can appear but in both cases degenerations of P, q are in some sense "balanced"; see supposition (1.4) below. (The functions P, q are considered in the form $P(x)=x^{1-\beta}$, $q(x)=x^{\beta}$, $\beta\in(0, 1)$, or $P(x)=x$, $q(x)=\ln(kx+1)$.

1. Notations and Preliminaries. Let us denote for brevity $I=(0, 1)$, $L=(0, 1\rangle$, $J=\langle 0, 1\rangle$. By $L_2=L_2(I)=L_2(J)$ we denote usual Lebesgue space with the norm

$$(1.1)\qquad \| u \|_2=\Big(\int_0^1 u^2(x)dx\Big)^{1/2}$$

and inner product

$$(1.2)\qquad (u, v)_2=\int_0^1 u(x)v(x)dx$$

(all functions following will be real ones).

By $C(I)(C(L), C(J))$ we denote the linear class of all continuous functions on I $(L, J$; $C(J)$ is linear space with supremum norm), $C^k(J)=\{u\in C(J);\ u^{(i)}\in C(J),$ $i=1, \dots, k\}$, $C^\infty(J)=\bigcap_{k=1}^{\infty}C^k(J)$ analogously for $C^k(I)$, $C^\infty(I)$, $C^k(L)$, $C^\infty(L)$; $C_0^\infty(I)=D(I)=\{u\in C^\infty(J);\ \text{support } u\subset I\}$. $\mathscr{D}'(I)$ is the space of Schwarz distributions. By $W^{k,p}(a, b)$ we denote the Sobolev space of all functions integrable on (a, b) with the power p together with their (distributional) derivatives up to the order k (k integer, $p\geq 1$), with the norm $\| u \|_{W^{k,p}(a,b)}$ $=(\Sigma_{i=0}^{k}\int_a^b | u^{(i)}(x)|^p\, dx)^{1/p}$; $W_0^{k,p}(a, b)=\overline{\mathscr{D}(a, b)}$.

The functions P, $q\in C^\infty(L)\cap C(J)$ are supposed to be positive on L and equal to zero at the origin (the maximal smoothness is not necessary and we suppose it is only for the sake of simplicity). By Q we denote an auxiliary function

$$(1.3)\qquad Q(x)=\int_x^1 \frac{1}{q(\xi)}d\xi,\quad x\in(0, 1\rangle.$$

This function can be bounded or unbounded but previously we will assume that the additional condition

$$(1.4)\qquad \lim_{x\to 0+} P(x)\,Q(x)=0$$

holds. By $L_{2;q}$ we denote the space of all functions u such that $uq^{1/2}\in L_2$, with the norm

$$(1.5)\qquad \| u \|_{2;q}=\| uq^{1/2} \|_2=\Big(\int_0^1 u^2 q dx\Big)^{1/2}$$

and with the inner product

$$(1.6)\qquad (u, v)_{2,q}=(uq^{1/2},\ vq^{1/2})_2=\int_0^1 uvq dx$$

analogously for $L_{2;P}$, $L_{2;1/P}$. The space $L_{2;q}$ is obviously the complete space and the mapping $u\to v=uq^{1/2}$ is an isometry between L_2, $L_{2;q}$, which conserve $\mathscr{D}(I)$; subsequently the space $\mathscr{D}(I)$, which is dense in L_2, is dense in $L_{2;q}$ as well. The same holds for $L_{2;P}$ $L_{2;1/P}$. Obviously

$$C(J) \frown L_{2;1/P} \frown L_{2,P} \tag{1.7}$$

with continuous embeddings (which is denoted by the sign $\frown$).

2. Sobolev Spaces with Weight. Functions P, q are supposed as in Section 1, i. e. $P,\ q \in C^\infty(L) \cap C(J)$, $P(0)=q(0)=0$, $P(x)>0$, $q(x)>0$ for $x>0$ and, moreover, the supposition (1.4) holds; Q is defined by (1.3).

Definition 2.1. Let us denote by W the space $W=\{u \in C(L);\ u(1)=0,\ (u/P)' \in L_{2;q}\}$ (here $(u/P)'$ is the distributional derivative) with the inner product

$$(u,\ v)_W = \left(\left(\frac{u}{P}\right)',\ \left(\frac{v}{P}\right)'\right)_{2;q} = \int_0^1 \left(\frac{u}{P}\right)'\left(\frac{v}{P}\right)' q\,dx \tag{2.2}$$

and with the corresponding norm

$$\| u \|_W = (u,\ u)_W^{1/2} = \left(\int_0^1 \left|\left(\frac{u}{P}\right)'\right|^2 q\,dx\right)^{1/2}. \tag{2.3}$$

By $\overset{\circ}{W}$ we denote the closure of $\mathscr{D}(I)$ with respect to this norm.

It is obvious that the inner product (2.2) is a non-negative bilinear form on W and hence (2.3) is a seminorm. To prove the implication $\| u \|_W=0 \Rightarrow u=0$, we establish the following simple estimate which yields a number of consequences:

Lemma 2.1. *Let $u \in W$; then we have*

$$\left|\frac{u(x)}{P(x)} - \frac{u(y)}{P(y)}\right| \leq Q^{1/2}(x)\left(\cdot\int_x^y \left|\left(\frac{u}{P}\right)'\right|^2 q\,dx\right)^{1/2}, \quad \forall x,\ y \in (0,\ 1\rangle,\ x<y \tag{2.4 a}$$

and

$$| u(x) | \leq P(x) Q^{1/2}(x) \| u \|_W. \tag{2.4 b}$$

Proof. We have $(u/P)(x) = -\int_x^y (u/P)'(\xi) q^{1/2}(\xi) q^{-1/2}(\xi)\,d\xi + (u/P)(y)$, and via Hölder inequality we obtain (2.4 a) which yields (2.4 b) by choice $y=1$.

Let us remark that, under supposition (1.4), the inequality (2.4 b) yields $\lim_{x\to 0+} u(x)=0$ and hence we can extend u up to the point 0.

Lemma 2.2. 1) *The expression* (2.3) *is the norm (and not the seminorm only).*

2) $\overset{\circ}{W} \frown W \frown C(J) \frown L_{2,1/P} \frown L_2$.

3) *The spaces W, $\overset{\circ}{W}$ are complete unitary spaces.*

Proof. The only nontrivial fact is the completeness of the space W. Hence, let $u_n \in W$ be a Cauchy sequence (in W). Thanks to (2.4 b) it has the Cauchy property in $C(J)$ as well, and there exists $u \in C(J)$, $u(1)=0$, such that $u_n(x) \rightrightarrows u(x)$ (uniformly on J). The completeness of $L_{2,q}$ guarantees the existence of the function $v = \lim_{n\to\infty} (u_n/P) \in L_{2,q}$. It is easy to see that $v=(u/P)'$ in the sense of distributions which proves the assertion 3.

In comparison with a usual Sobolev space without weight it is easy to show that the function $u \in W$ belongs to the space $W^{1,2}$ $(\varepsilon,\ 1)$ for an arbitrary $\varepsilon \in (0, 1)$ and the norm of this Sobolev space is equivalent to the norm of the type (2.3) but with integral taken over $(\varepsilon, 1)$. Supposing additionally $|Q(x)| \leq C$ for $x \in J$ and $P' \subset L_1(I)$, we can prove $W \subset W^{1,1}(I)$ but for Q unbounded this last embedding cannot be obtained without further restrictions on P, q (besides (1.4)). Of course, we have

Lemma 2.3. *Let $u \in W$; then $u' \in L_2(\varepsilon, 1)$ for all $\varepsilon \in I$. Moreover, denoting*

$$\| u \|_{W^\varepsilon} = \left(\int_\varepsilon^1 ((u/P)')^2 q dx \right)^{1/2},$$

we have

$$(2.5) \qquad \| u' \|_{L_1(\varepsilon,1)} \leq K_1(\varepsilon) \| u \|_{W^\varepsilon}; \quad K_1(\varepsilon) = \left(\int_\varepsilon^1 (P^2/q) dx \right)^{1/2} + \left(\int_\varepsilon^1 | P' | dx \right) Q^{1/2}(\varepsilon),$$

$$(2.6) \qquad \| u' \|_{L_2(\varepsilon,1)} \leq K_2(\varepsilon) \| u \|_{W^\varepsilon}; \quad K_2(\varepsilon) = 2[\max_{x \geq \varepsilon} (P^2/q) + \left(\int_\varepsilon^1 | P' |^2 dx \right) Q(\varepsilon)],$$

$$(2.7) \qquad \| u \|_{W^\varepsilon} \leq K_3(\varepsilon) \| u \|_{W^{2,1}(\varepsilon,1)}; \quad K_3(\varepsilon) = 2 \int_\varepsilon^1 (| P' |^2 + P^2) P^{-4} q dx.$$

Proof. Because of $P \in C^\infty(I)$ and $u = uPP^{-1}$ we obtain for the distributive derivative u' the expression $u' = (u/P)'P + (u/P)P'$ and, vice versa, $(u/P)' = u'P^{-1} - uP'P^{-2}$, from which we obtain easily the above estimates via Hölder inequality, writing, possibly, $1 = q^{1/2} q^{-1/2}$.

a)
$$\int_\varepsilon^1 | u' | dx = \int_\varepsilon^1 | Pq^{-1/2} (u/P)' q^{1/2} + P'(u/P) | dx$$
$$\leq \left[\left(\int_\varepsilon^1 Pq^{-1} dx \right)^{1/2} + \left(\int_\varepsilon^1 | P' | dx \right) \max_{x \geq \varepsilon} Q^{1/2}(x) \right] \| u \|_{W^\varepsilon}$$

b)
$$\int_\varepsilon^1 | u' |^2 dx \leq 2 \int_\varepsilon^1 (P^2 q^{-1} (u/P)'^2 q + P'^2 (u/P)^2) dx$$
$$\leq 2 \left[\max_{x \geq \varepsilon} P^2 q^{-1} + \int_\varepsilon^1 (P')^2 Q(\varepsilon) dx \right] \| u \|^2_{W^\varepsilon}$$

(using inequality $| P^{-1} u(x) | \leq Q^{1/2}(x) \| u \|_{W^\varepsilon}$, $\varepsilon \in (0, x)$, analogous to (2.4 b))

c)
$$\int_\varepsilon^1 (u/P)'^2 q dx \leq 2 \int_\varepsilon^1 [u'^2 P^{-2} + u^2 P^{-4} P'^2] q dx.$$

Now, if Q is bounded, then K_1 is bounded as well and $W \subset W^{1,1}$, but boundedness of K_2, K_3 requires other restrictions on P, q.

Now we can prove the following assertion:

Lemma 2.4. *The embedding $W \frown L_2$ is completely continuous.*

Proof. Let $M \subset W$ be a bounded subset of W. The (2.4 b) yields that it is bounded in $C(J)$, too. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that for arbitrary $f \in M$ holds $(\int_0^\delta f^2 dx)^2 < \varepsilon/4$. But (2.6) and (2.7) give $\| u \|_{W^\delta} \sim \| u \|_{W^{1,2}(\delta,1)}$ and thanks to the complete continuity of the embedding

$W^{1,2}(\delta, 1) \to L_2(\delta, 1)$ there exists a finite $\varepsilon/2$-net in the set M_δ of all functions from M restricted on $(\delta, 1)$ (in the L_2-norm). The estimate of $\int_0^\delta f^2 dx$ guarantees that these functions form a finite ε-net in the set M as a subset of $L_2(I)$ and hence M is precompact.

Summarizing, we can write down the following theorem:

Theorem 2.1 *(Embedding theorem). Suppose that for the functions P, q from $C^\infty(I) \cap C(J)$ with $P' \in L_1(I)$ the condition (1.4) holds. Then*

$$\mathring{W} \frown W \frown C(J) \frown L_{2;1/P} \frown L_2 \tag{2.8}$$

and the embedding $W \subset L_2$ is completely continuous. Moreover, if q^{-1} is integrable, then $W \frown W^{1,1}(I)$.

In the next part we consider the behaviour of $u \in W$ near the origin. First let us suppose that Q is bounded (and hence $Q(0)$ has a sense). Then the (2.4a) and (2.4b) enable us to establish

Lemma 2.5. *Let Q be a bounded function on J. Then the following statements hold:*

a) *The limit*

$$\lim_{x \to 0+} \frac{u(x)}{P(x)} \stackrel{\text{def}}{=} \frac{u}{P}(0) \tag{2.9}$$

("trace" of u/P) exists for every $u \in W$; moreover, the mapping T: $u \to (P^{-1}u)(0)$ is continuous mapping from W to R^1.

b) *If $u \in \mathring{W}$, then $(P^{-1}u)(0) = 0$.*

Proof. The first statement follows immediately from (2.4). To prove b) we write for $u \in \mathscr{D}(I)$

$$(P^{-1}u)(x) = \int_0^x (P^{-1}u)'(\xi) q^{1/2} q^{-1/2} d\xi,$$

which gives the estimate $|u(x)| \leq P(x)[Q(0) - Q(x)]^{1/2} \|u\|_W$. This estimate rests to be true (by continuity) for $u \in \mathring{W}$, which proves the assertion.

Obviously the function PQ belongs to W, but in virtue of statement b) it does not belong to $\mathring{W}$. Hence, if Q is bounded, then $\mathring{W}$ is a proper subset of W.

The statement b) can be converted. At first we establish an auxiliary lemma which will be also useful in consideration of the case of Q unbounded.

Lemma 2.6. *Let $u \in W$ and let there exists for arbitrary $\varepsilon \in I$ a positive $\delta < \varepsilon$, and the function $\varphi_\varepsilon \in C(J)$ with the following properties:*

(i) $$0 \leq \varphi_\varepsilon(x) \leq 1, \quad x \in J,$$

(ii) $$\varphi_\varepsilon(x) = 0 \text{ for } x \in (0, \delta), \ \varphi_\varepsilon(x) = 1 \text{ for } x \in (\varepsilon, 1), \ \varphi'_\varepsilon \in C(\langle \delta, \varepsilon \rangle),$$

(iii) $$\lim_{\varepsilon \to 0+} \int_\delta^\varepsilon q(P^{-1}u\varphi'_\varepsilon)^2 dx = 0.$$

Then $u_\varepsilon = u\varphi_\varepsilon \to u$ in W.

Proof. Obviously $u_\varepsilon - u \in W$ and $(P^{-1}u_\varepsilon)' = (P^{-1}u)'\varphi_\varepsilon + P^{-1}u\varphi'_\varepsilon$ (in the sense of distributions). Let us estimate the norm $\|u_\varepsilon - u\|_W$. We have

$$\|u-u_\varepsilon\|_W=\|(P^{-1}u)'(1-\varphi_\varepsilon)-P^{-1}u\varphi_\varepsilon'\|_{2;q}\leqq\|(P^{-1}u)'(1-\varphi_\varepsilon)\|_{2,q}+\|P^{-1}u\varphi_\varepsilon'\|_{2;q}$$
$$=\left(\int_0^\varepsilon(P^{-1}u)'^2(1-\varphi_\varepsilon)^2qd\xi\right)^{1/2}+\left(\int_\delta^\varepsilon(P^{-1}u\varphi_\varepsilon')^2qd\xi\right)^{1/2},$$
from which the lemma follows. Now we can prove

Lemma 2.7. *Let $u\in W$ has "zero trace" $(P^{-1}u)(0)=0$. Then $u\in\mathring{W}$.*

Proof. First, there is possible, under the assumptions of the Lemma 2.7, to construct the functions φ_ε satisfying conditions (i)—(iii) from Lemma 2.6 (independently on $u\in\mathring{W}$) such that $\varphi_\varepsilon'(x)=K/x$ for $x\in(\delta,2\delta)$, $\delta=\varepsilon/2$. Of course, then $\varphi_\varepsilon(x)=K\ln x+C$ on $(\delta,2\delta)$, and for validity of (i), (ii) it is sufficient that $\varphi_\varepsilon(2\delta)-\varphi_\varepsilon(\delta)=K\ln 2=1$. On the other hand,

$$\int_\delta^\varepsilon(P^{-1}u)^2q\varphi_\varepsilon'^2\leqq C\max_{x\leqq\varepsilon}(P^{-1}u)(x)\int_\delta^\varepsilon\xi^2d\xi=C_1\max_{x\leqq\varepsilon}(P^{-1}u)(x)\to 0.$$

Hence we have $u_\varepsilon\to u$ in W, $u_\varepsilon=0$ on the neighbourhood of the origin. Now, obviously $u\in W_0^{1,2}(\delta,1)$ and hence it can be arbitrarily approximate by a function $\varphi\in\mathscr{D}(\delta,1)$ in the norm of $W^{1,2}(\delta,1)$ (see e. g. [4]), and hence, in virtue of Lemma 2.1, in the norm $\|f\|_{W^\delta}$ as well. But functions u_ε, φ equal to zero on $(0,\delta)$ and hence $\|u_\varepsilon-\varphi\|_{W^\delta}=\|u_\varepsilon-\varphi\|_W$, and by usual diagonal choice argument we obtain the lemma.

Lemma 2.7 together with the assertion b) of Lemma 2.5 completely characterize the space $\mathring{W}$. Taking in consideration that $PQ\in W$, we can write W as a direct sum of $\mathring{W}$ and the linear hull of PQ. Thus we obtain the

Corollary 2.1. *Let Q be a bounded function. Then the set $C^\infty(L)\cap W$ is dense in W.*

Now let us consider the case of unbounded Q. Then the above arguments concerning traces of $(P^{-1}u)$ fail. Moreover, we have

Lemma 2.8. *Let Q be an unbounded function. Then $W=\mathring{W}$.*

Proof. We shall construct functions φ_ε, $\delta(\varepsilon)$, which satisfy conditions of Lemma 2.6 for arbitrary $u\in W$; the rest of the proof is the same as in Lemma 2.7. To this end, let us put $\varphi_\varepsilon'(x)=K/Qq=-K(\ln Q)'$ for $x\in(\delta,\varepsilon)$. Conditions (i), (ii) will be fulfilled, if $1=K\int_\delta^\varepsilon\varphi(\xi)d\xi$, which gives $K=-1/[\ln Q]_\delta^\varepsilon$. Now let be $u\in W$ and let us denote $V=u/P$. We have

$$\int_\delta^\varepsilon V^2q\varphi_\varepsilon'^2d\xi=K^2\int_\delta^\varepsilon Q^{-1}V^2(Qq)^{-1}d\xi\leqq\|u\|_W K$$

(thanks to (2.4 a)). Now $\lim_{x\to 0^+}Q(x)=+\infty$ and hence it is possible to find $\delta=\delta(\varepsilon)$ such that K tends to zero, if $\varepsilon\to 0$, q. e. d.

Eventually we can summarize

Theorem 2.2 *(Trace theorem). Let P, q be such that Q is bounded. Then for every $u\in W$ it exists $(P^{-1}u)(0)=\lim_{x\to 0^+}(u(x)/P(x))$, which depends continuously on u from W. Moreover, $u\in W$ belongs to $\mathring{W}$ iff $(u(0)/P(0))=0$.*

Theorem 2.3 *(Density theorem). If the function Q is unbounded, then $\mathscr{D}(I)$ is dense in W. If it is bounded, then $W-\mathring{W}\ni PQ$, $W=\mathring{W}\oplus[PQ]$, and the class $C^\infty(L)\cap W(J)$ is dense in W.*

3. Duals to Sobolev Spaces with Weight. The spaces W, $\mathring{W}$ are normed linear spaces and hence we can define their duals.

Definition 3.1. Let X, $\mathring{X}$ be the spaces dual to W, $\mathring{W}$, respectively, with the pairing $\langle v, u\rangle$, $u \in W$, $(\mathring{W})$, $v \in X$, $(\mathring{X})$.

To derive properties of these spaces, we repeat properties of an adjoint operator:

Lemma 3.1. *Let E, F be Banach spaces with duals E', F' and let B be a continuous linear operator from the dense subset $E_0 \subset E$ to F. Then the adjoint operator $\tilde{B}$, defined by $\langle \tilde{B}v, u\rangle_{E,E'} = \langle v, Bu\rangle_{F,F'}$, $u \in E_0$, $v \in F'$, is also a continuous operator from F' to E', which is one-to-one iff the range of B is dense in F, and, vice versa, it has dense range iff B is one-to-one. Moreover, B is completely continuous iff B is* (Schauder theorem, see e. g. [3, p. 390]).

Theorem 2.1 asserts that there exist continuous embeddings i_0: $\mathring{W} \to L_2$, i: $W \to L_2$, which have obviously dense range and hence we can embed the space L_2', dual to L_2, into X as well as into $\mathring{X}$ with embeddings j, j_0 adjoint to i, i_0 (which represents restrictions of functionals on L_2'). Identifying by usual way L_2 with its dual, we obtain embeddings $W \frown L_2 \frown X$, $\mathring{W} \frown L_2 \frown X$, all completely continuous; the embeddings $L_2 \subset X$, $L_2 \subset \mathring{X}$ are realized by the formula

$$\langle u, v\rangle = (u, v)_0 = \int_0^1 u(x)v(x)dx.$$

The space $\mathring{X}$ can be embedded to the space $\mathscr{D}'(I)$ of distributions and the space $\mathscr{D}(I)$ is dense in $\mathring{X}$. Though it is dense in X as well, this space in the case of Q bounded cannot be identified with $\mathring{X}$ and nevertheless with any subspace of $\mathscr{D}'(I)$, because $\mathscr{D}(I)$ is not dense in W. In this case every functional on W can be restricted to the functional on $\mathring{W}$, but this restriction has nontrivial kernel, generated by "Trace functional" T: $u \to u/P(0)$ (which follows from trace Theorem 2.2 and from the decomposition of W from Theorem 2.2). (In both cases the space X contains the functional $\Delta_\xi : \langle u, \Delta_\xi\rangle = u(\xi)$, which forms natural initial condition for the physical problem mentioned in the introduction, and this element is an extension of the Dirac distribution, but if Q is bounded, this extension is not unique.)

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