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ON THE EXISTENCE, UNIQUENESS AND CHARACTERIZATION OF PERFECT SPLINES OF MINIMUM NORM

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Summary. Existence and a new characterization of perfect splines of mininum norm, related to totally positive kernels, are obtained by a unified method applicable to the class of monotone norms (norms, for which $|f(x)| \leqq |g(x)|$ implies $\|f\| \leqq \|g\|$). The method is based on the duality between this problem and best L^1-approximation, which provides a pointwise improvement theorem for perfect splines. Similar results are obtained for norms induced by inner-products, by the equivalence between this case and the self-dual case of perfect splines of minimum L^1-norm.

As in the case of the L^p-norms the knots and zeros of the minimal perfect splines determine best tensor-product approximations to totally positive kernels in norms which are a tensor-product of a monotone norm and the weighted L^1-norm. Also n-widths and optimal spaces in the sense of Kolmogorov and Gelfand can be obtained via the minimal perfect splines, for classes of functions defined by integral operators with totally positive kernels.

1. Introduction. Perfect splines of minimum norm are of central importance in the theory of n-width for classes of functions related to integral operators with totally positive (TP) kernels, and also in optimal tensor product approximations to such kernels.

Given a TP kernel $K(x, y) \in C([a, b] \times [c, d])$, a perfect spline with n knots $c = y_0 < y_1 < \ldots < y_n < y_{n+1} = d$ is defined as

$$\varphi_{\mathbf{y}}(x) = \int_c^d K(x, y) h_{\mathbf{y}}(y)\, dy\,;\ h_{\mathbf{y}}(y) = (-1)^i h(y),\ y_i \leqq y \leqq y_{i+1},\ 0 \leqq i \leqq n, \tag{1.1}$$

with $h \in L^\infty[c, d]$, $h > 0$ in $[c, d]$.

Existence of perfect splines of minimum L^p-norms, $1 \leqq p < \infty$, leading to various n-width results, is proved in [10] for TP kernels satisfying certain additional assumptions on the independence of their sections. Also optimal tensor product approximations to the kernel K in norms of the form

$$|||f(x, y)||| = [\int_a^b [\int_c^d |f(x, y)|\, dy]^p\, dx]^{1/p} \tag{1.2}$$

are obtained in [11]. The case $p = \infty$ is investigated in [10], while the case $p = 2$ with assumptions of different nature on the kernel K is analysed in [9].

The present work provides a unified method for the derivation of the existence and a characterization of perfect splines of minimum norm. The method we use is independent of the explicit form of the norm and applies to the wide class of monotone norms (norms, for which $|f| \leq |g|$ on $[a, b]$ implies $\|f\| \leq \|g\|$). Consequently also n-widths results and tensor product optimal approximations to bivariate functions can be obtained for monotone norms.

Our method of proof is based on the duality between the problem of perfect splines of minimum norm and best L^1-approximation. This duality is the key to an "improvement theorem" for perfect splines, in analogy to the known "improvement theorems" for monosplines [6, 11]. The knots of the perfect spline of minimum norm are characterized as a fixed point in R^n of an "improving" transformation, based on the canonical points for best L^1-approximation [3]. The "improving" transformation is used to prove the uniqueness of perfect splines of minimum L^1-norm for kernels of the form $K(x, y) = f(x-y)$ and for $h(y) \equiv 1$ [1].

For extended TP (ETP) kernels this approach yields the existence and a characterization of perfect splines of minimum norm for all monotone norms. For TP kernels such results are obtained only for various sub-classes of monotone norms, containing the class of L^p-norms, $1 \leq p \leq \infty$, depending on the smoothness and the independence of the sections $K(\cdot, y)$, $K(x, \cdot)$ of the kernels. Since the method of proof is the same for all these cases, we present it for the simpler case of ETP kernels and all monotone norms. The more general case of TP kernels is treated in [2].

The self-dual case of perfect splines of minimum L^1-norm related to symmetric kernels, with its specific structure, provides a tool in the analysis of perfect splines of minimum norm for norms induced by inner-products. The latter case is proved to be equivalent to the former under certain conditions on the kernel and the inner-product.

The knots and zeros of perfect splines of minimum monotone norms determine best tensor product approximations $\Sigma_{i=1}^{n} u_i(x) v_i(y)$ to ETP kernels $K(x, y)$ in norms of the form

$$\|\|f\|\| = \int_c^d \|f(\cdot, y)\| h(y)\, dy, \tag{1.3}$$

where $\|\cdot\|$ is a monotone norm, and $h \in L^\infty[c, d]$, $h > 0$ [2].

Also n-widths of Kolmogorov and Gelfand type are determined by perfect splines of minimum norm for two classes of functions:

(i) $$K_h = \{\int_c^d K(x, y)\sigma(y)\, dy : \sigma \in L^\infty[c, d], |\sigma(y)| \leq h(y),\ y \in [c, d]\}$$

with the n-widths measured by monotone norms or by inner-product norms

(ii) $$K_X = \{\int_c^d K(x, y) g(y)\, dy : g \in X,\ \|g\| < 1\},$$

where $X \subset L^1[c, d]$ is normed by a monotone norm $\|\cdot\|$, and the n-widths are measured by weighted L^1-norms. All these n-widths are equal to the norms of minimal perfect splines with respect to the appropriate norm and the corresponding optimal spaces are related either to the set of knots or to the set of zeros of these minimal perfect splines [2].

2. Perfect Splines of Minimum Norm. For $K \subset C^{1,0}([a, b] \times [c, d])$ ETP of order 2 in x and order 1 in y (for definitions of terms related to total positivity consult [5]), we consider the minimum of the function

$$F(\mathbf{y}) = \| \varphi_{\mathbf{y}} \| = \| \int_c^d K(\cdot, y) h_{\mathbf{y}}(y) \, dy \| \tag{2.1}$$

in

$$S^n [c, d] = S^n = \{ \mathbf{y} \in R^n : c \leq y_1 < \ldots < y_n \leq d \} \tag{2.2}$$

for two classes of norms: the monotone norms and inner-product norms.

$F(\mathbf{y})$ is continuous in S^n whenever $\| \cdot \|$ is monotone and can be extended continuously to $\bar{S}^n$ by defining $F(\mathbf{y}) = \| \varphi_{\mathbf{z}} \|$, with $\mathbf{z} \in S^k$, $k \leq n$, consisting of points among $y_1, \ldots, y_n$ with odd multiplicities. Hence $F(\mathbf{y})$ has a minimum in $\bar{S}^n$. That this minimum is attained in S^n only, can be concluded from the following result:

Theorem 2.1 (Improvement theorem for perfect splines). *Let* $\mathbf{z} \in S^k$, $k \leq n$. *Then there exist n points* $c < y_1 < \ldots < y_n < d$ *such that*

$$\| \varphi_{\mathbf{y}} \| < \| \varphi_{\mathbf{z}} \|; \tag{2.3}$$

whenever $k < n$, or if $k = n$ and a non-zero best approximation to $\varphi_{\mathbf{z}}$ from the span of $K(\mathbf{z}) = \{ K(x, z_i), \ i = 1, \ldots, n \}$ exists.

The proof of this result, which is central to this work, is based on the following observations:

(a) $\varphi_{\mathbf{z}}$ is in the convexity cone of the T-system $K(\mathbf{z})$.

(b) For any best approximation u to $\varphi_{\mathbf{z}}$ from the span of $K(\mathbf{z})$, the error $\varphi_{\mathbf{z}} - u$ has k zeros counting multiplicities up to order 2, due to the monotonicity of the norm [8].

(c) Let

$$K[\boldsymbol{\xi}] = \{ \frac{\partial^{l_i}}{\partial x^{l_i}} K(\xi_i, y), \ l_i = \begin{cases} 0 & \xi_i > \xi_{i-1}, \\ 1 & \xi_i = \xi_{i-1} \end{cases} \ i = 1, \ldots, k \} \tag{2.4}$$

with $\boldsymbol{\xi} = (a \leq \xi_1 \leq \ldots \leq \xi_k \leq b)$ the zeros of $\varphi_{\mathbf{z}} - u$. Then

$$| (\varphi_{\mathbf{z}} - u)(x) | = \int_c^d | K(x, y) - v_x(y) | \, h(y) \, dy, \tag{2.5}$$

where $v_x \in \operatorname{Span} K[\boldsymbol{\xi}]$ interpolates $K(x, \cdot)$ at $z_1, \ldots, z_k$.

(d) For $\tilde{\boldsymbol{\xi}} \in S^n$ containing all the zeros of $\varphi_{\mathbf{z}} - u$ ($\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}$ if $k = n$), the canonical signature $h_{\mathbf{y}}$, $\mathbf{y} \in S^n$, satisfying

$$\int_c^d u(y) h_{\mathbf{y}}(y) \, dy = 0, \ u \in K[\tilde{\boldsymbol{\xi}}] \tag{2.6}$$

corresponds to $\mathbf{y} \neq \mathbf{z}$, whenever $k < n$ or $k = n$ and $u \neq 0$.

(e) By the relation of the canonical signature to best L^1-approximation, and since $K(x, \cdot)$ is in the convexity cone of $K [\tilde{\boldsymbol{\xi}}]$

$$\int_c^d K(x, y) h_{\mathbf{y}}(y) \, dy \,| = \inf \{ \int_c^d | K(x, y) - v(y) | \, h(y) \, dy | \, v \in \operatorname{Span} K [\tilde{\boldsymbol{\xi}}] \}, \tag{2.7}$$

which in view of (2.5) leads to the relation

(2.8) $$|\varphi_{\mathbf{y}}(x)| \leq |(\varphi_{\mathbf{z}}-u)(x)|, \quad x\in[a, b],$$

with equality only at the zeros of $\varphi_{\mathbf{z}}-u$.

(f) Any monotone norm has the property [7]: $|f(x)| \leq |g(x)|$, $x\in[a,b]$ with equality only at the zeros of g implies $\|f\|<\|g\|$. Hence by (2.8) and by the definition of u

$$\|\varphi_{\mathbf{y}}\| < \|\varphi_{\mathbf{z}}-u\| \leq \|\varphi_{\mathbf{z}}\|.$$

As direct consequences of Theorem 2.1 we obtain

Corollary 2.1. *Let $\varphi_{\mathbf{y}^*}$, $\mathbf{y}^*\in S^m$, $m\leq n$, be a perfect spline of minimum norm*

(2.9) $$\|\varphi_{\mathbf{y}^*}\| \leq \|\varphi_{\mathbf{y}}\|, \quad \mathbf{y}\in S^k, \ k\leq n.$$

Then $m=n$ and $c<y_1^*<\dots<y_n^*<d$. Moreover, $\varphi_{\mathbf{y}^*}$, $\mathbf{y}^*\in S^n$, has property (2.9) if and only if

(2.10) $$\|\varphi_{\mathbf{y}^*}\| < \|\varphi_{\mathbf{y}} - \sum_{i=1}^{k} a_i K(x, y_i)\|, \quad \mathbf{y}\in S^k, \ \sum_{i=1}^{k} a_i^2>0, \quad k\leq n.$$

Corollary 2.2. *A point $\mathbf{y}^*\in S^n$ has property (2.9) only if $\mathbf{y}^*$ is a fixed point of the transformation mapping $\mathbf{z}\in S^n$ into the canonical points $\mathbf{y}\in S^n$ of the T-system $K[\xi]$, corresponding to the zeros $\xi=(a\leq\xi\leq\dots\leq\xi_n\leq b)$ of the error in best approximating $\varphi_{\mathbf{z}}$ by elements from the span of $K(\mathbf{z})$ in the monotone norm $\|\cdot\|$.*

All these results hold for the L^p-norms $1\leq p\leq\infty$ also in case $K\in C([a, b]\times[c, d])$ is strictly TP, since in the proof of Theorem 2.1 $\varphi_{\mathbf{z}}-u$ has simple zeros.

For the L_g^1-norm: $\|u\|_{1,g} = \int_a^b |u(x)|\, g(x)\,dx$, $g>0$, $g\in L^\infty[a, b]$,

the transformation in Corollary 2.2 is of the form $T_h T_g$, where T_g (T_h) is the transformation mapping $\mathbf{y}\in S^n[c, d]$ ($\mathbf{x}\in S^n[a, b]$) into the point $T_g\mathbf{y}\in S^n[a, b]$ ($T_h\mathbf{x}\in S^n[c, d]$), corresponding to the canonical points for the T-system $K(\mathbf{y})$ ($K[\mathbf{x}]$) and the weight function g (h). It is possible to show by a method, similar to that used in the proof of the uniqueness of the algebraic monosplines of minimum L^1-norm [4], that the transformations T_g, T_h are contracting in case $K(x, y)=f(x-y)$ and $g\equiv1$, $h\equiv1$ [1]. This together with Corollary 2.2 yields

Theorem 2.2. *There exists a unique perfect spline of minimum L_g^1-norm in case $g\equiv1$, $h\equiv1$ and $K(x, y)=f(x-y)$.*

The self adjoint case corresponding to the transformation $T_gT_h=T_h^2$, namely to a symmetric kernel $K(x, y)=K(y, x)$, defined on $[a, b]^2$, with $h\equiv g$, has a specific structure:

Theorem 2.3. *Any perfect spline*

(2.11) $$\varphi_{\mathbf{y}^*} = \int_a^b K(\cdot, y)\, h_{\mathbf{y}^*}(y)\,dy$$

of minimum L_h^1-norm has n simple zeros at his knots $y_1^*, \dots, y_n^*$, whenever $K(x, y)=K(y, x)$ and $[c, d]=[a, b]$.

This structure of the perfect spline (2.11) plays a central role in the derivation of an analogous result to Corollary 2.1 for perfect splines of minimum inner-product norms. In fact it can be shown that for an inner product $(\,,)$ and a kernel $K(z, y)$, defined on a domain $D_1 \times D_2$ with $[a, b] \subset D_2$, such that

$$G(x, y) = (K(\cdot, x), K(\cdot, y)) \tag{2.12}$$

is continuous and strictly TP on $[a, b]^2$, the following problems are equivalent:

$$\min\{\| \int_a^b G(x, y)\, h_{\mathbf{y}}(y)\, dy \|_{1,h} : \mathbf{y} \in S^k[a, b],\ k \leq n\}, \tag{2.13}$$

$$\min\{(\int_a^b K(\cdot, y) h_{\mathbf{y}}(y)\, dy,\ \int_a^b K(\cdot, y)\, h_{\mathbf{y}}(y)\, dy) : \mathbf{y} \in S^k[a, b],\ k \leq n\}. \tag{2.14}$$

This equivalence is stated more precisely in the following

Theorem 2.4. *A perfect spline* $\varphi_{\mathbf{y}^*} = \int_a^b K(\cdot, y)\, h_{\mathbf{y}^*}(y)\, dy$ *is a solution of problem* (2.14) *if and only if the perfect spline* $\varphi_{\mathbf{y}^*} = \int_a^b G(\cdot, y)\, h_{\mathbf{y}^*}(y)\, dy$ *is a solution of problem* (2.13). *Moreover, if* $\varphi_{\mathbf{y}^*}$ *is of minimum norm, then* $(\varphi_{\mathbf{y}^*}, \varphi_{\mathbf{y}^*}) = \| \varphi_{\mathbf{y}^*} \|_{1,h}$ *and*

$$(\varphi_{\mathbf{y}^*}, \varphi_{\mathbf{y}^*}) \leq (\varphi_{\mathbf{y}} - \sum_{i=1}^k a_i K(\cdot, y_i),\ \varphi_{\mathbf{y}} - \sum_{i=1}^k a_i K(\cdot, y_i)),\ \ \mathbf{y} \in S^k[a, b], k \leq n, \tag{2.15}$$

with a strict inequality whenever $\Sigma_{i=1}^k a_i^2 > 0$.

Theorems 2.4 and 2.2 imply the uniqueness of the perfect spline of minimum inner-product norm corresponding to $h = 1$, if the kernel G in (2.12) is of the form $f(x-y)$.

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