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ON EQUICONVERGENT MATRIX TRANSFORMS OF POWER SERIES

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Summary. Let A be an arbitrary summability-matrix and $\{\omega_n(z)\}$ the sequence of the A-transforms of $\{z^n\}$. It is shown that there always exists a regular generalized Riesz-matrix (R, p, M), such that the sequences $\{\tau_n(z)\}$ of the (R, p, M)-transforms of $\{z^n\}$ and $\{\omega_n(z)\}$ are almost equiconvergent outside a given simply connected domain.

1. Introduction and Statement of Results. Let $A=(\alpha_{n\nu})n=0,1,\dots;$ $\nu=0,1,\dots$ be a matrix with complex coefficients and let $f(z)=\Sigma_{\nu=0}^{\infty}a_\nu^f z^\nu$ be any power series with the radius of convergence $R^f>0$. The A-transforms of this series are given by $\sigma_n^f(z):=\Sigma_{\nu=0}^{\infty}\alpha_{n\nu}s_\nu^f(z)$, where $s_\nu^f(z):=\Sigma_{\mu=0}^{\nu}a_\mu^f z^\mu$.

The method A is called p-regular (or regular for power series), if for all those power series the sequence $\{\sigma_n^f(z)\}$ converges compactly to $f(z)$ in the disk $D_{R^f}:=\{z:|z|<R^f\}$. It has been shown by Luh [4], that A is p-regular if and only if the following conditions hold:

$$\lim_{n\to\infty}\alpha_{n\nu}=0 \qquad \text{for } \nu=0,1,\dots;$$

$$\lim_{n\to\infty}\sum_{\nu=0}^{\infty}\alpha_{n\nu}=1;$$

$$\sup_n\sum_{\nu=0}^{\infty}|\alpha_{n\nu}|r^\nu<\infty \quad \text{for all } r\in(0,1).$$

It is obvious that a regular method is p-regular, but, conversely, a p-regular method needs not to be regular in general.

In this note we deal with a generalization of the well-known Riesz method. Let $p=\{p_\nu\}$ be a fixed sequence of complex numbers and let $P_n:=\Sigma_{\nu=0}^{n}p_\nu$. Suppose $M=\{m_n\}$ is a subsequence of natural numbers, for which $P_{m_n}\neq 0$ for $n=0,1,\dots$ Then the (R,p,M)-method is defined by the rowfinite matrix $A=(\alpha_{n\nu})$ with the elements

$$\alpha_{p_\nu}=p_\nu/P_{m_n}\ (0\leq\nu\leq m_n);\ \alpha_{n\nu}=0\ (\nu>m_n).$$

A Riesz method (R,p,M) is regular if and only if

$$\lim_{n\to\infty}P_{m_n}=\infty;\qquad \sup_n|P_{m_n}|^{-1}\cdot\sum_{\nu=0}^{m_n}|p_\nu|<\infty.$$

Riesz methods (R,p,M) were introduced by Faulstich [1].

A classical problem in summability theory is the construction of matrices $A=(\alpha_{n\nu})$, such that a given power series is compactly A-summable on certain "large" sets. It is well-known, that for this purpose it suffices to study the behaviour of the A-transforms of the geometric sequence $\{z^n\}$, which are given by $\omega_n(z):=\Sigma_{\nu=0}^{\infty}\alpha_{n\nu}z^\nu$. Using Okada's theorem or its refinement by Gawronski and Trautner [2], results for general power series are easily obtained.

In a lot of papers summability methods have been constructed, which sum the geometric sequence on certain prescribed sets (cf. [1, 5, 6, 7], where further references are given). However, the methods, which have been obtained there, are p-regular but not regular in general. Thus the question arises, if even regular methods can be constructed, which have similar properties.

In this paper we shall show that if there exists a matrix $A=(\alpha_{n\nu})$ such that $\omega_n(z)=\Sigma_{\nu=0}^{\infty}\alpha_{n\nu}z^\nu$ is well-behaved on a certain set $M\subset\boldsymbol{C}$, then there always exists a regular (R, p, M)-method such that $\tau_n(z):=P_{m_n}^{-1}\cdot\Sigma_{\nu=0}^{m_n}p_\nu z^\nu$ is also well-behaved in the same sense almost on M.

To be precise we prove the following

Theorem. *Suppose* $\omega_n(z):=\Sigma_{\nu=0}^{\infty}\alpha_{n\nu}z^\nu$ *has radius of convergence* ρ_n *and let* $\rho:=\inf\{\rho_n>1:n\geqq 0\}$. *Let* G_0 *be a simply connected domain with* $1\notin G_0$, $\boldsymbol{D}:=\{z:|z|<1\}\subset G_0$ *and let* Γ *be a Jordan-arc with* $1\in\Gamma$, $\infty\in\Gamma$ *which does not intersect* $G_0\cup\bar{\boldsymbol{D}}\setminus\{1\}$. *Then there exists a regular* (R,p,M)-*method with the following properties*:

(1) $\{\tau_n(z)\}$ *converges compactly to zero on* $G_0\cup\bar{\boldsymbol{D}}\setminus\{1\}$.

(2) $\{\tau_n(z)-\omega_n(z)\}$ *converges compactly to zero on* $(G_0\cup\bar{\boldsymbol{D}}\cup\Gamma)^c\cap\boldsymbol{D}_\rho$ *and on* $(\Gamma\setminus\{1\})\cap\boldsymbol{D}_\rho$.

2. Proof of the Theorem. a) We use some notations: Suppose K is a compact set with $1\notin K$, which contains the origin in its interior $\mathring{K}$. Let $\gamma:=\operatorname{dist}(0,K^c)$, $\delta:=2(1-\gamma)$ and

$$M(K):=\{z:1<|z|<1+\delta,\ \operatorname{dist}(z,K^c)<\delta\}\ \cup\{z:|z-1|<\delta\}.$$

According to Tomm [7] there are two sequences $\{K_n\}$ and $\{L_n\}$ of compact sets, such that $(K_n\cup L_n)^c$ is connected and the following conditions hold:

(3) $0\in\mathring{K}_n\subset K_n\subset G_0$, $L_n\cap(K_n\cup\bar{\boldsymbol{D}})=\emptyset$ for all n.

(4) Every compact set $K\subset G_0\cup\bar{\boldsymbol{D}}\setminus\{1\}$ is contained in $(K_n\cup\bar{\boldsymbol{D}})\setminus M(K_n)$ for all sufficiently large n.

(5) Every compact set $K\subset\Gamma\setminus\{1\}$ is contained in L_n for all sufficiently large n.

(6) Every compact set $K\subset(G_0\cup\bar{\boldsymbol{D}}\cup\Gamma)^c$ is contained in L_n for all sufficiently large n.

b) We construct a sequence of polynomials $\{z^{g_k}Q_k(z)\}$. Let $Q_0(z)\equiv 1$, $g_0=0$ and suppose that for $n\in\boldsymbol{N}$ the polynomials $z^{g_0}Q_0(z),\ldots,z^{g_{n-1}}Q_{n-1}(z)$ have already been determined. By g_n we denote the degree of the polynomial $z^{g_{n-1}}Q_{n-1}(z)$.

We first chose $s_n\in\boldsymbol{N}$ such that the polynomial $\omega_n^{(s_n)}(z):=\Sigma_{\nu=0}^{s_n}\alpha_{n\nu}z^\nu$ satisfies the inequality

(7) $$\max_{\underline{D}_{\rho-(n)^{-1}}} |\omega_n(z)-\omega_n^{(s_n)}(z)|<n^{-1}.$$

Next we can find a polynomial $Q_n(z):=\Sigma_{\lambda=g_n}^{g_{n+1}-1} a_\lambda z^{\lambda-g_n}$ with the following properties:

(8) $$Q_n(1)=\sum_{\lambda=g_n}^{g_{n+1}-1} a_\lambda=1,$$

(9) $$\sum_{\lambda=g_n}^{g_{n+1}-1} |a_\lambda|<2,$$

(10) $$\max_{(K_n\cup\bar{D})\setminus M(K_n)} |Q_n(z)|<(\max_{K_n\cup\bar{D}} |z^{g_n}|)^{-1}\cdot 2^{-n},$$

(11) $$\max_{L_n}|Q_n(z)-z^{-g_n}[(n+1)\omega_n^{(s_n)}(z)-\sum_{k=0}^{n-1} z^{g_k}Q_k(z)]|<(\max_{L_n}|z^{g_n}|)^{-1}.$$

The existence of $Q_n(z)$ follows from [7, Theorem 2.1]. By induction we obtain the sequence $\{z^{g_k}Q_k(z)\}$.

c) Let $K\subset G_0\cup\bar{D}\setminus\{1\}$ be any compact set. By (4) we have $K\subset(K_n\cup\bar{D})\setminus M(K_n)$ for all sufficiently large n and from (10) we conclude

$$\max_K |z^{g_n}Q_n(z)|\leqq \max_{(K_n\cup\bar{D})\setminus M(K_n)} |z^{g_n}Q_n(z)|<2^{-n}.$$

Thus the series $\Sigma_{k=0}^\infty z^{g_k}Q_k(z)$ converges compactly on $G_0\cup\bar{D}\setminus\{1\}$. The function $\pi(z):=\Sigma_{k=0}^\infty z^{g_k}Q_k(z)$ is analytic on G_0 and its power series $\pi(z)=\Sigma_{\nu=0}^\infty p_\nu z^\nu$ has radius of convergence $R\geqq 1$. For $m_n:=g_{n+1}-1$ we obtain by the special form of the polynomials $z^{g_k}Q_k(z)$

$$\sum_{\nu=0}^{m_n} p_\nu z^\nu=\sum_{\nu=0}^{g_{n+1}-1} p_\nu z^\nu=\sum_{k=0}^{n}\sum_{\lambda=g_k}^{g_{k+1}-1} a_\lambda z^\lambda=\sum_{k=0}^{n} z^{g_k}Q_k(z).$$

By (8) we have $P_{m_n}:=\Sigma_{\nu=0}^{m_n} p_\nu=\Sigma_{k=0}^n Q_k(1)=n+1$ and therefore we obtain $R=1$. According to (9) we have

$$\sum_{\nu=0}^{m_n} |p_\nu|=\sum_{k=0}^{n}\sum_{\lambda=g_k}^{g_{k+1}-1} |a_\lambda|<2(n+1).$$

d) Let us now consider the (R,p,M)-method, which is defined by the sequences $p=\{p_\nu\}$ and $M=\{m_n\}$. We have $|P_{m_n}|^{-1}\Sigma_{\nu=0}^{m_n}|p_\nu|<(n+1)^{-1}\cdot 2(n+1)=2$ and the regularity of (R,p,M) follows. The (R,p,M)-transforms of $\{z^n\}$, which are given by

$$\tau_n(z)=P_{m_n}^{-1}\sum_{\nu=0}^{m_n} p_\nu z^\nu=(n+1)^{-1}\sum_{k=0}^{n} z^{g_k}Q_k(z),$$

are compactly convergent to zero on $G_0\cup\bar{D}\setminus\{1\}$.

Let $K\subset(\Gamma\setminus\{1\})\cap \boldsymbol{D}_\rho$ be a compact set. By (5) and (11) we obtain for all sufficiently large n

$$\max_K|\tau_n(z)-\omega_n^{(s_n)}(z)|=\max_K|(n+1)^{-1}\sum_{k=0}^{n} z^{g_k}Q_k(z)-\omega_n^{(s_n)}(z)|$$

$$\leq(n+1)^{-1}\max_{L_n}|z^{g_n}\{Q_n(z)-z^{-g_n}[(n+1)\omega_n^{(s_n)}(z)-\sum_{k=0}^{n-1}z^{g_k}Q_k(z)]\}|$$

$$\leq(n+1)^{-1}\max_{L_n}|z^{g_n}|.(\max_{L_n}|z^{g_n}|)^{-1}=(n+1)^{-1}.$$

According to (7) the sequence $\{\tau_n(z)-\omega_n(z)\}$ converges compactly to zero on $(\Gamma\setminus\{1\})\cap\boldsymbol{D}_\rho$.

By (6) it follows analogously that $\{\tau_n(z)-\omega_n(z)\}$ converges compactly to zero on $(G_0\cup\overline{\boldsymbol{D}}\cup\Gamma)^c\cap\boldsymbol{D}_\rho$.

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