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ON QUASI-HERMITE-FEJÉR INTERPOLATION: POINTWISE ESTIMATES

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Summary. We give a brief survey of the results obtained by numerous authors in the so-called Quasi-Hermite-Fejér-Interpolation (QHFI) and deal mainly with new quantitative assertions. These are based upon more general theorems which yield estimates involving the first order modulus of continuity of the first derivative, its least concave majorant, or the second order modulus of continuity; classical ω_1-statements and inequalities, involving the least concave majorant of $\omega_1(f, \cdot)$, are also a consequence of the general assertions. The proofs are based upon the technique of 'double smoothing'. The applications include sequences of positive and nonpositive QHFI operators as well.

1. Introduction and Results. The image of a function $f \in C[-1, 1]$ under a QHFI operator $Q_n: C[-1, 1] \to \Pi_{2n+1}$ is the uniquely determined algebraic polynomial satisfying for a given sequence of nodes $1 = x_0 > x_1 > \cdots > x_n > x_{n+1} = -1$ the $2n+2$ conditions

$$Q_n(f, x_k) = f(x_k),\ 0 \leqq k \leqq n+1, \text{ and } (Q_n f)'(x_k) = 0, \quad 1 \leqq k \leqq n.$$

For an arbitrary choice of nodes the linear operator Q_n has the form

$$Q_n(f, x) = f(-1)\frac{1-x}{2w(-1)^2}\, w(x)^2 + f(1)\frac{1+x}{2w(1)^2}\, w(x)^2$$

$$+ \sum_{k=1}^{n} f(x_k)\, \frac{1-x^2}{1-x_k^2}\, [1 + c_k(x - x_k)]\, \Big(\frac{w(x)}{w'(x_k)\,(x-x_k)}\Big)^2,$$

where $w(x) = c \prod_{k=1}^{n} (x - x_k)$, $c \neq 0$, and $c_k(x) = \frac{2x_k}{1-x_k^2} - \frac{w''(x_k)}{w'(x_k)}$, $1 \leqq k \leqq n$.

Because of the position of the nodes $x_1, \ldots, x_n \in (-1, 1)$ it is natural to investigate QHFI processes based on the two end points ± 1 and the roots $x_1, \ldots, x_n$ of Jacobi polynomials $P_n^{(\alpha, \beta)}$, $\alpha, \beta > -1$.

The systematic approach to this kind of approximation process can be traced to E g e r v á r y, T u r á n [V30]*, who proved that the sequence $Q_n(f, \cdot) = Q_n^{(0,0)}(f, \cdot)$, constructed with the aid of the triangular matrix consisting

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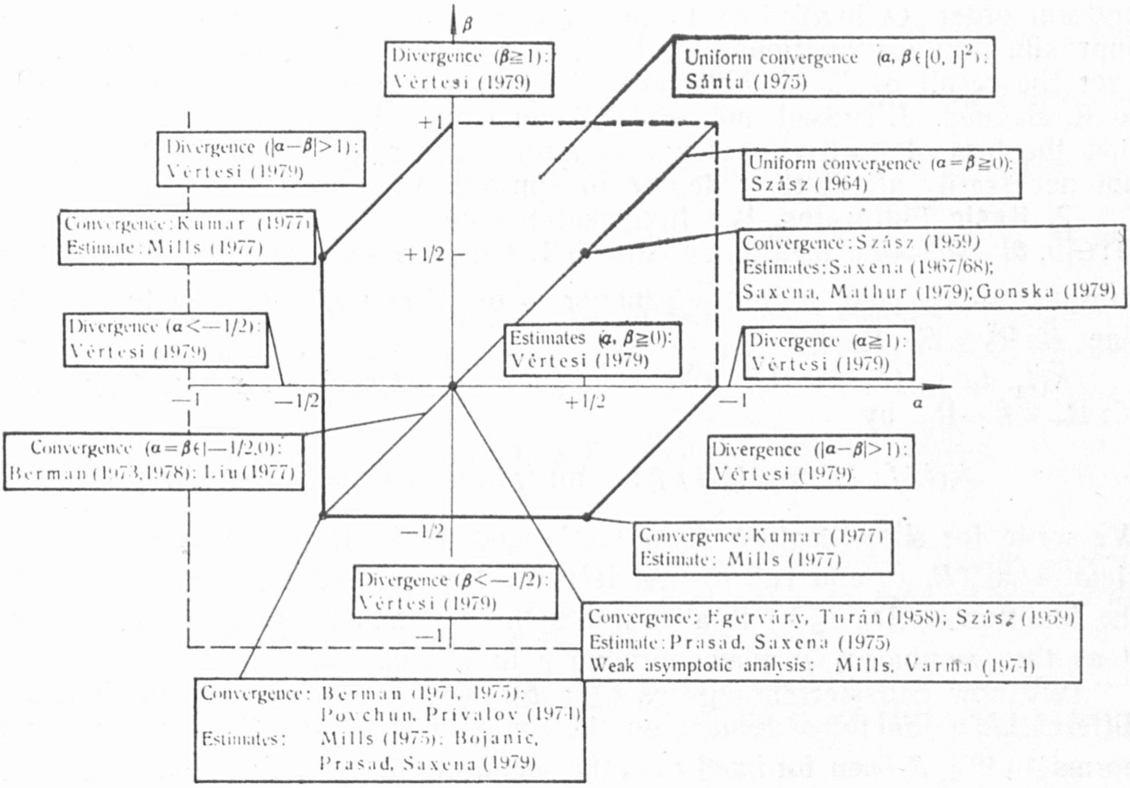
* For the sake of brevity all papers referred to in V é r t e s i's article [22] are not included in our references. Thus [Vn], $n \in N$ means the n-th reference in that paper.

of the roots of the Legendre polynomials $P_n^{(0,0)}$, converges uniformly in $[-1, 1]$ to f whenever $f \in C[-1, 1]$. This work has been continued by numerous authors. In order to give a brief survey of the known results we choose to represent them in the diagram given below. The resulting



For uniform convergence inside the "half-open" hexagon see [22, Theorem 3.3] The points (0, 1) and (1, 0) belong to the dotted line (no uniform convergence in general)

hexagon encloses those values of (α, β), for which $Q_n^{(\alpha,\beta)}(f, \cdot)$ converges uniformly to f for all $f \in C[-1, 1]$. For values of (α, β), with $\alpha, \beta > -1$, not enclosed in the hexagon, there is no uniform convergence in general. Further information on the literature and on special approximation processes are given for instance in Vértesi's paper [22] and in the forthcoming work of Knoop [12].

The known quantitative theorems for the cases $\alpha, \beta \geqq 0$ (see Vértesi [22]) and $\alpha = \beta = -1/2$ (see Mills [V28] and Bojanic, Prasad, Saxena [6]) cover the approximation of arbitrary continuous functions only, using mainly the first order modulus of continuity of the function under consideration.

It is our main aim to prove — with the aid of much more general estimates in terms of a certain K-functional Ω — pointwise improvements and pointwise modifications of these results involving for instance a first order modulus of the first derivative $\omega_1(f', \cdot)$ or the second order modulus of continuity $\omega_2(f, \cdot)$.

The proofs will be given for the cases $\alpha=\beta=1/2$ (positive operators) and $\alpha=\beta=-1/2$ (non-positive operators) only. This will show clearly that the underlying general techniques apply in the positive and non-positive case as well. The general results will imply in both cases for instance that any $f\in C^1[-1, 1]$ with $f'\in \mathrm{Lip}\,\delta$, $0<\delta\leq 1$, can be approximated with the uniform order $O((\ln n)^{1-\delta}/n)$. In the case $\alpha=\beta=-1/2$ the application for the approximation for functions $f\in C[-1, 1]$ will give a remarkable improvement over the result of T. M. Mills and yield an estimate similar to the one due to R. Bojanic, J. Prasad and R. B. Saxena. In particular, it can be seen that the loss of positivity of the operators occurring for $\min(\alpha, \beta)<0$, does not necessarily affect their degree of approximation in a negative way.

2. Basic Estimates. We first sketch how to construct the functional $\Omega : C[a, b]\times \mathbf{R}_+^3\to\mathbf{R}_+$ mentioned above. Let E denote a real vector space, U a subspace of E, and p and $\bar{p}$ seminorms on E and U, respectively. We define $\tilde{K}:\mathbf{R}_+^2\times E\to\mathbf{R}_+$ by

$\tilde{K}(t_1, t_2, f; (E, p), (U, \bar{p})):=\inf\{p(f-g)+t_1p(g)+t_2\bar{p}(g):g\in U\}$, and $K: \mathbf{R}_+\times E\to\mathbf{R}_+$ by

$$K(t, f; (E, p), (U, \bar{p})):=\inf\{p(f-g)+t\bar{p}(g): g\in U\}.$$

We write for simplification $\tilde{K}(t_1, t_2, f)$ and $K(t, f)$ respectively, if it is clear what (E, p) and $(U, \bar{p})$ are. It is readily verified that for fixed (t_1, t_2) the functional $\tilde{K}(t_1, t_2, \cdot)$ is a seminorm on E; thus it makes sense to use it as the seminorm $\bar{p}$ when defining a functional $K(t, \cdot)$.

We now consider the spaces $C^i[a, b]$, $i\in\{1, 2\}$, of i-times continuously differentiable functions defined on the finite interval $[a, b]$, with the seminorms $\|\cdot^{(i)}\|_\infty$.* Then for fixed t_1, t_2 the seminorm $\tilde{K}(t_1, t_2, f; (C^1[a, b], \|\cdot^{(1)}\|_\infty), (C^2[a, b], \|\cdot^{(2)}\|_\infty))$ may be used in the definition of

$$\Omega(f; t, t_1, t_2):=K(t, f; (C[a, b], \|\cdot\|_\infty), (C^1[a, b], \tilde{K}(t_1, t_2, \cdot))),$$

$t\geq 0$, $f\in C[a, b]$. Information on the magnitude of Ω is given in

Theorem 2.1. *Let Ω be defined as above. Then the following inequalities are true for any $(f; t, t_1, t_2)\in C[a, b]\times\mathbf{R}_+^3$, $0<h\leq b-a$:*

$$\Omega(f; t, t_1, t_2)\leq\begin{cases} (1+\frac{t}{h})\,\omega_1(f, h), & \text{(i)}\\ tt_1\cdot\|f'\|_\infty+(t+\frac{tt_2}{h})\,\omega_1(f', h), & \text{(ii)}\\ (\frac{3}{2}+\frac{2tt_2}{h^2})\omega_2(f, h)+\frac{2tt_1}{h}\,\omega_1(f, h), & \text{(iii)}\\ \frac{1}{2}\,\omega_1^*(f, 2t), & \text{(iv)}\\ tt_1\,.\|f'\|_\infty+t\omega_1^*(f', 2t_2), & \text{(v)}\end{cases}$$

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* Here $^{(i)}$ denotes i-fold differentiation.

where, for $k=1, 2$, $\omega_k(f, \cdot)$ *denotes the k-th order modulus of continuity of* f, $\omega_1^*(f, \cdot)$ *is the least concave majorant of* $\omega_1(f, \cdot)$ *and* (ii) *and* (v) *are valid for* $f \in C^1[a, b]$ *only.*

For the proofs we refer to our investigation in [7] and [8] and to two papers by Peetre [16, 17].

As shown in [8], for certain linear operators we have the following estimate in terms of Ω.

Theorem 2.2. *Let* $L: C[a, b] \to C[a, b]$ *be continuous and linear, satisfying for all* $x \in [a, b]$

(i) $|L(f_1, x)-f_1(x)| \leq \varphi(x) \cdot \|f_1'\|_\infty$ *for all* $f_1 \in C^1[a, b]$ *with a function* $\varphi \geq 0$ *and*

(ii) $|L(f_2, x)-f_2(x)| \leq \gamma_1(x) \cdot \|f_2\|_\infty + \gamma_2(x) \cdot \|f_2''\|_\infty$ *for all* f_2 *in* $C^2[a, b]$ *with* $\gamma_1, \gamma_2 \geq 0$.

Moreover, we suppose for all $x \in [a, b]$ *that the quotients* $\gamma_i(x)/\varphi(x)$, $i=1, 2$, *are finite. Then for every* $f \in C[a, b]$ *and every* $x \in [a, b]$ *we have the inequality*

$$|L(f,x)-f(x)| \leq (\|L\|+1)\Omega\left(f; \frac{\varphi(x)}{\|L\|+1}, \frac{\gamma_1(x)}{\varphi(x)}, \frac{\gamma_2(x)}{\varphi(x)}\right).$$

For certain positive linear operators we obtain (see [8]).

Theorem 2.3. *Let* $L: C[a, b] \to C[a, b]$ *be a positive linear operator satisfying* $L(e_0, \cdot)=e_0$, *where* e_i *denotes the i-th monomial. Then for any* $f \in C[a, b]$ *and all* $x \in [a, b]$ *the estimate*

$$|L(f, x)-f(x)| \leq 2\Omega\left(f; \frac{L(|e_1-x|, x)}{2}, \frac{|L(e_1-x, x)|}{L(|e_1-x|, x)}, \frac{L((e_1-x)^2, x)}{2L(|e_1-x|, x)}\right)$$

holds.

Theorems 2.1, 2.2 and 2.3 will be the general tools for proving the above mentioned statements.

Remark 2.4. In view of Theorem 2.3 we would like to remark that 'one half' of the uniformly convergent QHFI processes constructed with the aid of Jacobi roots consist of sequences of positive operators, namely the ones for $\alpha, \beta \in [0, 1)$. Thus for the remaining cases Theorem 2.3 is not applicable; however, it will be seen in Ch. 4 how the desired results can be achieved as well.

3. Pointwise Estimates for $\alpha=\beta=1/2$. In this case the operators have the form (see Szász [V10])

$$Q_n(f, x)=f(-1)\frac{1-x}{2(n+1)^2}U_n^2(x)+f(1)\frac{1+x}{2(n+1)^2}U_n^2(x)$$

$$+\sum_{k=1}^{n} f(x_k)\frac{(1-x^2)(1-x_k x)}{(n+1)^2}\left(\frac{U_n(x)}{x-x_k}\right)^2,$$

where U_n is the n-th Čebyšev polynomial of the second kind. Since Q_n is a positive linear operator for all $n \geq 1$, Theorem 2.3 will be the main tool for proving the following

Theorem 3.1. *For the operators* Q_n, *based upon the roots of the Čebyšev polynomials of the second kind, we have for all* $n \geq 1$, *all* $f \in C[-1, 1]$ *and every* $x \in [-1, 1]$:

(i) $|Q_n(f, x) - f(x)| \leq c\,\omega_1(f, \frac{1+\sqrt{1-x^2}\,.\ln(n+1)}{n+1})$.

Provided that f is in $C^1[-1, 1]$ *we have in addition*

(ii) $|Q_n(f, x) - f(x)| \leq \frac{c}{n+1}\{(1-x^2)|U_n(x)U_{n-1}(x)| \cdot \|f'\|_\infty$

$$+s_n(x)\omega_1(f', (1-x^2)\,U_n^2(x)/s_n(x))\},$$

where $s_n(x) := 1+\sqrt{1-x^2}\,.\ln(n+1)$.

(iii) $|Q_n(f, x) - f(x)| \leq 4\{\omega_2(f, \sqrt{1-x^2}\,.|U_n(x)|/\sqrt{n+1})$

$$+[\sqrt{1-x^2}\,.|U_{n-1}(x)|/\sqrt{n+1}\,]\omega_1(f, \sqrt{1-x^2}\,.|U_n(x)|/\sqrt{n+1})\}.$$

c always denotes a strictly positive real number independent of f, x and n.

Proof. In order to apply Theorem 2.3 one has to evaluate Q_n for the functions $|e_1-x|$, $(e_1-x)^2$ and e_1-x. This can be done by using Theorem 1 in the paper of Saxena, Mathur [V 13] and some properties of the polynomials U_n (see e. g. Schönhage [21]). For details see [7]. Theorem 2.3 then gives an estimate in terms of Ω, and the remaining part of the proof follows from Theorem 2.1.

Remark 3.2. It would also have been possible to give estimates involving $\omega_1^*(f, \cdot)$ or $\omega_1^*(f', \cdot)$. This was omitted since such estimates are only interesting when determining optimal constants. This problem is dealt with in our paper [9] for instance.

By proper choices of h (cf. Theorem 2.1) the same type of proof leads to the uniform estimates in

Theorem 3.3. *For the operators* $Q_n = Q_n^{(1/2,\ 1/2)}$ *the following inequalities are valid* ($n \geq 1$, $f \in C[-1, 1]$ *or* $C^1[-1, 1]$, *respectively*):

(i) $\|Q_n f - f\|_\infty \leq c\omega_1(f, \ln(n+1)/n+1)$,

(ii) $\|Q_n f - f\|_\infty \leq \frac{c}{n+1}\{\|f'\|_\infty + \ln(n+1)\omega_1(f', 1/\ln(n+1))\}$,

(iii) $\|Q_n f - f\|_\infty \leq 4\{\omega_2(f, 1/\sqrt{n+1}) + (1/\sqrt{n+1})\omega_1(f, 1/\sqrt{n+1})\}$.

c denotes a constant not depending on f and n.

Corollary 3.4. *If* $f \in C^1[-1, 1]$, *having a derivative* $f' \in \mathrm{Lip}\,\delta$, $0 < \delta \leq 1$, *then Theorem* 3.3 (ii) *implies* $\|Q_n f - f\|_\infty = O(\frac{(\ln(n+1))^{1-\delta}}{n+1})$.

4. Pointwise Estimates for $\alpha = \beta = -1/2$. In this case the operators are of the form (see e. g. Povchun, Privalov [V 11]):

$$Q_n(f, x) = f(-1)\frac{(1-x)\,T_n^2(x)}{2} + f(1)\frac{(1+x)\,T_n^2(x)}{2}$$

$$+\sum_{k=1}^{n} f(x_k)\frac{1-x^2}{1-x_k^2}(1+xx_k-2x_k^2)\frac{T_n^2(x)}{n^2(x-x_k)^2},$$

where T_n is the n-th Čebyšev polynomial. Since the operators Q_n are not positive, the analogy of Theorem 3.1 will be obtained from Theorems 2.2 and 2.1.

Theorem 4.1. *The QHFI operators Q_n based upon the roots of Čebyšev polynomials satisfy the following inequalities* ($n \geq 1$, $f \in C[-1, 1]$ *or* $C^1[-1, 1]$, *respectively*, $x \in [-1, 1]$):

(i) $|Q_n(f, x) - f(x)| \leq c\omega_1(f, \frac{1+\sqrt{1-x^2} . \ln n}{n})$,

(ii) $|Q_n(f, x) - f(x)| \leq \frac{c}{n} \{|T_n(x)| . \|f'\| + r_n(x)\omega_1(f', \frac{T_n^2(x)}{r_n(x)})\}$,

where $r_n(x) := 1 + \frac{2}{\pi} T_n^2(x)\sqrt{1-x^2} . \ln n$,

(iii) $|Q_n(f, x) - f(x)| \leq c\{\omega_2(f, \frac{|T_n(x)|}{\sqrt{n}}) + \frac{1}{\sqrt{n}}\omega_1(f, \frac{|T_n(x)|}{\sqrt{n}})\}$.

Here c always denotes a positive real constant independent of f, x and n, but depending upon the upper bound of $(\|Q_n\|)_{n \geq 1}$.

Proof. As shown by Mills [14], $(Q_n)_{n \geq 1}$ is a sequence of uniformly bounded linear operators. Thus $\|Q_n\| \leq A$ for all $n \geq 1$. For the proof of (i) in Theorem 2.2 we will use the method employed by Liu [13] and Berman [4] among others writing

$$Q_n(f, x) - f(x) = H_n(f, x) - f(x) + T_n^2(x)(a_n x + b_n), \text{ where}$$

$$a_n = \frac{1}{2}[(f(1) - H_n(f, 1)) - (f(-1) - H_n(f, -1))], \text{ and}$$

$$b_n = \frac{1}{2}[(f(1) - H_n(f, 1)) + (f(-1) - H_n(f, -1))];$$

here H_n denotes the ordinary Hermite-Fejér process for Čebyšev nodes. Thus we can estimate $|Q_n(f, x) - f(x)|$ by

$$|H_n(f, x) - f(x)| + T_n^2(x)[|f(-1) - H_n(f, -1)| + |f(1) - H_n(f, 1)|].$$

Applying the result of V. G. Amelkovič [1] (see also [10]) we obtain for any function $f_1 \in C^1[-1, 1]$

$$|Q_n(f_1, x) - f_1(x)| \leq \frac{1}{n}(c + \frac{2}{\pi} . T_n^2(x) . \sqrt{1-x^2} . \ln n) . \|f_1'\|_\infty =: \varphi_n(x) . \|f_1'\|_\infty.$$

For the proof of (ii) in Theorem 2.2 we apply the Berman idea again, use the positivity of H_n, the fact that $H_n e_0 = e_0$ and Lemma 2 of Minkova [15] to arrive first at ($f_2 \in C^2[-1, 1]$, $x \in [-1, 1]$):

$$|H_n(f_2, x) - f_2(x)| \leq |H_n(e_1 - x, x)| . \|f_2'\|_\infty + \frac{1}{2} H_n((e_1 - x)^2, x) . \|f_2''\|_\infty$$

$$= \frac{1}{n}|T_n(x) T_{n-1}(x)| . \|f_2'\|_\infty + \frac{1}{2n} . T_n^2(x) . \|f_2''\|_\infty$$

(see [7] aud [18]). This leads to

$$| Q_n(f_2, x)-f_2(x)| \leq \frac{3}{n} | T_n(x)| . \| f_2' \|_\infty + \frac{3}{2n} T_n^2(x) . \| f_2'' \|_\infty$$

$$=: \gamma_{1,n}(x) . \| f_2' \|_\infty + \gamma_{2,n}(x) . \| f_2'' \|_\infty .$$

Theorem 2.2 gives the inequality

$$| Q_n(f, x)-f(x)| \leq (\| Q_n \| + 1)\, \Omega\left(f, \frac{\varphi_n(x)}{\| Q_n \| + 1}, \frac{\gamma_{1,n}(x)}{\varphi_n(x)}, \frac{\gamma_{2,n}(x)}{\varphi_n(x)}\right),$$

and the rest of the proof is easily carried out with the aid of Theorem 2.1.

Remark 4.2. Statements analogous to the ones in Remark 3.2, Theorem 3.3, and Corollary 3.4 hold in this case, too.

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* The reader is reminded of the fact that this list of references is not complete since we only intend to give a supplement to the compilation in Vértesi's paper [22].

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