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ONE-SIDED APPROXIMATION OF THE SOLUTION OF A TWO-POINT BOUNDARY VALUE PROBLEM BY CUBIC SPLINES

G. R. Grozev, S. M. Markov

Summary. We approximate the solution $y=y(x)$ of the two-point boundary value problem $y''-q(x)y=f(x)$, $y(0)=y(1)=0$, by means of two explicitly constructed cubic splines $\underline{s}(x)$ and $\overline{s}(x)$ on a uniform mesh $\{x_i=ih\}_{i=-1}^{n+1}$ such that $\underline{s}(x)\leqq y(x)\leqq \overline{s}(x)$ on $[0, 1]$ and $\underline{s}(0)=\underline{s}(1)=0$, $\overline{s}(0)=\overline{s}(1)=0$. It is shown that an upper bound for the error of the constructed approximation is i) $O(\omega(f; h)+\omega(q, h))$ in the situation when the parameters $f(x)$ and $q(x)$ are continuous, ii) $O(h)$ when f, q possess bounded derivatives and iii) $O(h^2)$ when f, q possess bounded second derivatives.

1. Introduction. We consider the two-point boundary value problem

$$
(1)\qquad \begin{aligned} & y''(x)-q(x)y(x)=f(x), \quad x\in[0, 1], \\ & y(0)=y(1)=0, \end{aligned}
$$

where $q(x)\geqq 0$, $f(x)<0$ are continuous on $[0, 1]$.

Remark. The restriction $q(x)\geqq 0$ arises from the physical meaning of the two-point boundary value problem and the restriction $f(x)<0$ from the boundedness of the right-hand side in (1), which is also a natural assumption.

We denote $Ly(x)=y''(x)-q(x)y(x)-f(x)$, $\Gamma y=(y(0), y(1))$, so that problem (1) can be written in the form

$$
\begin{aligned} & Ly(x)=0 \quad \text{on} \quad [0, 1], \\ & \Gamma y=0. \end{aligned}
$$

In what follows we shall denote the solution of (1) by $y(x)$.

The solution $y=y(x)$ of problem (1) monotonically depends on the parameters $q(x), f(x)$ and on the boundary conditions [1, 4, 5]. Taking this monotonicity into account, the following lemma can be proved:

Lemma 1. *Let* $u, y\in C^2[0, 1]$ *and* y *be solution of* (1).

i) *If* $0\leqq Lu(x)\leqq\delta$, $x\in[0, 1]$, $\Gamma u=0$, *then*

$$
-\delta/8\leqq u(x)-y(x)\leqq 0, \quad x\in[0, 1],
$$

$$
\|u-y\|_{L_1}\leqq\delta/12;
$$

ii) *If* $-\delta \le Lu(x) \le 0$, $x \in [0, 1]$, $\Gamma u = 0$, *then*

$$0 \le u(x) - y(x) \le \delta/8, \quad x \in [0, 1],$$
$$\| u - y \|_{L_1} \le \delta/12.$$

2. Construction of One-Sided Approximations of $y = y(x)$. We shall denote by $\underline{s}(x)$ and $\bar{s}(x)$ two cubic splines on the mesh $\{x_i = ih\}_{i=-1}^{n+1}$, such that $\underline{s}(x) \le y(x) \le \bar{s}(x)$. The following theorem suggests an algorithm for the construction of $\underline{s}(x)$, $\bar{s}(x)$:

Theorem 1. *Let* q, f *be continuous and* y *be solution of* (1). *Let* n *be a positive integer and* $h = 1/n$. *There exists* $\delta = \delta(h) > 0$, *such that:*

i) *The cubic spline* $\underline{s}(x)$, *determined* (*in a unique way*) *by the system*

$$\begin{cases} \underline{s}''(x_i) - q(x_i)\underline{s}(x_i) - f(x_i) = \delta, \ x_i = ih, \ i = 0, 1, \ldots, n, \\ \Gamma \underline{s} = 0 \end{cases}$$

satisfies on $[0, 1]$ *the inequalities* $0 \le \underline{s}''(x) - q(x)\underline{s}(x) - f(x) \le 2\delta$, $x \in [0, 1]$ (*which in view of Lemma 1 immediately imply* $-\delta/4 \le \underline{s}(x) - y(x) \le 0$, $x \in [0, 1]$, $\| \underline{s} - y \|_{L_1} \le \delta/6$);

ii) *The cubic spline* $\bar{s}(x)$, *determined* (*uniquely*) *by the system*

$$\begin{cases} \bar{s}''(x_i) - q(x_i)\bar{s}(x_i) - f(x_i) = -\delta, \ x_i = ih, \ i = 0, 1, \ldots, n, \\ \Gamma \bar{s} = 0, \end{cases}$$

satisfies on $[0, 1]$ *the inequalities* $-2\delta \le \bar{s}''(x) - q(x)\bar{s}(x) - f(x) \le 0$, $x \in [0, 1]$ (*which in view of Lemma 1 yields* $0 \le \bar{s}(x) - y(x) \le \delta/4$, $x \in [0, 1]$, $\| \bar{s} - y \|_{L_1} \le \delta/6$;

iii) $\delta = O(\omega(q; h) + \omega(f; h))$;

iv) *Moreover, if* q *and* f *possess bounded derivatives on* $[0, 1]$, *then* $\delta = O(h)$;

v) *If* q *and* f *possess bounded second derivatives on* $[0, 1]$, *then* $\delta = O(h^2)$.

Remark. The expressions $\delta = \delta(h) = \delta(q, f; h)$ are explicitly found [4] and involve bounds of the parameters q, f (and q', f' as the case may be). This allows for the construction of a numerical algorithm based on Theorem 1, which produces cubic splines with a prescribed one-sided approximation.

3. The Numerical Algorithm. In what follows we describe the construction of the upper one-sided approximation, since the lower approximation is constructed in a similar way.

Assume that we wish to construct a cubic spline $\bar{s}(x)$ with the property $0 \le \bar{s}(x) - y(x) \le d$, $x \in [0, 1]$, where d is a given positive number. We then proceed as follows:

1) Determine h such that $\delta(f, q; h) \le 4d$ and denote the value of $\delta(f, q; h)$, calculated for this particular value of h again by δ, so that $\delta \le 4d$.

2) Solve the tridiagonal system

$$\begin{cases} \bar{s}''(x_i) - q(x_i)\bar{s}(x_i) - f(x_i) = -\delta, \quad i = 0, 1, \ldots, n, \\ \bar{s}(0) = \bar{s}(1) = 0 \end{cases}$$

for the determination of the spline function $\bar{s}(x)$. Then, according to Theorem 1, we have $0 \le \bar{s}(x) - y(x) \le \delta/4 \le d$.

The realization of the described algorithm is straightforward. This algorithm can be also easily realized in computer arithmetic with directed roundings. Such an approach can provide for the validity of the inequalities $\underline{s}(x)\leq y(x)\leq \overline{s}(x)$ after the computer realization of the alogorithm, that is the inequalities cannot be damaged by the round-off error.

4. Best One-Sided Approximations of $y=y(x)$ by Cubic Splines. Denote by S_{Σ_n} the set of all cubic splines on the uniform mesh $\Sigma_n=\{ih\}_{i=0}^{n}$, $h=1/n$. Following Collatz [2, 3], consider the linear optimization problem with continuum number of constraints: $\overline{\delta}=\min\{\delta : s\in S_{\Sigma_n}, -\delta\leq L\overline{s}(x)\leq 0$ on $[0, 1]$, $\Gamma\overline{s}=0\}$.

From the above considerations it follows that for the best upper one-sided approximation $\overline{\varepsilon}$ by cubic splines of S_{Σ_n}:

$$\overline{\varepsilon}=\min\{\|\overline{s}(x)-y(x)\|_C : \overline{s}\in S_{\Sigma_n}, y(x)\leq\overline{s}(x) \text{ on } [0, 1]\}$$

we have

$$\overline{\varepsilon}\leq\overline{\delta}/8=\begin{cases} O(\omega(f;h)+\omega(q;h)), \text{ if } f, q \text{ are continuous,} \\ O(h), \text{ if } f, q \text{ possess bounded derivatives,} \\ O(h^2), \text{ if } f, q \text{ possess bounded second derivatives.} \end{cases}$$

Analogous results hold true for the best lower one-sided approximations.

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Centre for Mathematics and Mechanics
1090 Sofia, P. O. Box 373 Bulgaria

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