КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'81
София, 1983, с. 357—367

CONSTRUCTIVE FUNCTION THEORY'81
Sofia, 1983, p. 357—367

A CONSTRUCTIVE CHARACTERISTIC OF THE BEST ALGEBRAIC APPROXIMATION IN $L_p[-1,1]$ $(1\leqq p\leqq\infty)$

K. G. Ivanov

Summary. The problem to characterize the best algebraic approximation in a finite interval is of importance for the approximation theory. While the theory of the best trigonometric approximation has reached a high level of completeness, big difficulties due to the effect of the ends appear in the solution of the corresponding algebraic problems. Some characterizations of the best algebraic approximation are given by Džafarov [1], Potapov [4] and Stens, Wehrens [8]. All of them are based on the modified translation concept and the corresponding modified derivative concept. Fuksman [5] characterizes the best uniform algebraic approximation in terms of the usual derivative and the local modulus of continuity. Using new moduli, defined by means of the usual translation operator, in this paper we extend our results in [2, 6] to complete the characterization of the best algebraic approximation in $L_p(1\leqq p\leqq\infty)$.

1. Definitions and Denotations. We shall consider functions belonging to the spaces $L_p[a,b]$ $(1\leqq p\leqq\infty)$ with the norm

$$\|f\|_{L_p[a,b]}=\{\int_a^b|f(x)|^p dx\}^{1/p}\quad\text{for}\quad 1\leqq p<\infty;\quad \|f\|_p=\|f\|_{L_p[-1,1]}.$$

As usual when these problems are considered, $L_\infty[a,b]$ denotes the set $C[a,b]$ with sup norm. If f is a function of several variables, then the variable on which the norm is taken will be marked after the norm sign, e. g. $\|f(x,y)\|_{(x)L_p[a,b]}$

$$W_p^k[a,b]=\{f:f^{(k)}\in L_p[a,b]\}\quad\text{for}\quad k\in N=\{1,2,\ldots\}.$$

Let H_n be the set of all algebraic polynomials of a degree at most n. If $w\in C[-1,1]$, $w\geqq 0$, then the best L_p approximation of f by polynomials of degree n with the weight w is the quantity

$$E_n(w;f)_p=\inf\{\|w(f-Q)\|_p:Q\in H_n\};\quad E_n(1;f)_p=E_n(f)_p.$$

Henceforth k and n will be natural numbers. With c we denote absolute positive constants and with $c(A,B,\ldots)$ — constants depending only on the marked parameters. These constants may differ at each occurrence. Let us note that the values of the explicitly given constants are not the best possible ones.

For $x\in[-1,1]$ we set $\Delta(\delta, x)=\delta\sqrt{1-x^2}+\delta^2$, $\Delta_n(x)=\Delta(n^{-1}, x)$ ($\delta=\text{const}$). As a characteristic of the best algebraic approximation we shall use the moduli

$$\tau_k(f, w;\delta)_{p',p}=\| w(\cdot)\omega_k(f,\cdot;\ \delta(\cdot))_{p'}\|_p, \tag{1.1}$$

where the local $L_{p'}$ moduli ω_k are defined by

$$\omega_k(f, x;\delta(x))_{p'}=[(2\delta(x))^{-1}\int_{-\delta(x)}^{\delta(x)}|\Delta_v^k f(x)|^{p'}dv]^{1/p'}. \tag{1.2}$$

Here $1\leq p, p'\leq\infty$ and the finite difference $\Delta_v^k f(x)$ is defined as

$$\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}f(x+rv),\quad \text{if}\quad x,\ x+kv\in[-1,1]$$

and as 0 otherwise. In (1.1) and (1.2) δ is arbitrary positive function, but we shall consider only the cases $\delta(x)=d=\text{const}$ and $\delta=\Delta(d)$. The weight w is a non-negative continuous function. Usually w will belong to one of the following classes: $W=\{w\in C[-1,1]: w>0;\ w(x)\leq c(\lambda)w(t)$ for each $x, t\in[-1,1], |x-t|\leq\lambda\Delta_n(x)\}$; $W(s)=\{w\in W: w(x)\leq c\lambda^s w(t)$ for each $x, t\in[-1,1]$, $|x-t|\leq\lambda\Delta_n(x), \lambda\geq1\}$ for $s\geq0$; W_k is the set of all $w\in W$ satisfying the condition: there is $c(k, w)$ such that for each $m\leq n$ and for every $Q\in H_m$ the inequality

$$\| wQ^{(k)}(n\Delta_n)^k\|_p\leq c(k, w)m^k\| wQ\|_p$$

holds true. Let us note that: if $s=\max\{|\alpha|, |\beta|, |\gamma_1|, \dots, |\gamma_r|\}$, then

$$(\sqrt{1-x}+n^{-1})^\alpha(\sqrt{1+x}+n^{-1})^\beta\prod_{i=1}^{r}(|x-x_i|+n^{-1})^{\gamma_i}\in W(s);$$

$w_{n,\gamma}(x)=(n\Delta_n)^\gamma\in W(|\gamma|)$; $w_{n,|\gamma|}\in W_k$ for each $k\in\boldsymbol{N}$; if $cw_1\leq w_2\leq cw_1$ and $w_1\in W(s)$ (or W_k), then $w_2\in W(s)$ ($w_2\in W_k$); if $w_1\in W(s_1)$, $w_2\in W(s_2)$, then $w_1w_2\in W(s_1s_2)$.

In the following theorem we give some of the properties of τ_k (see [6, 7]).

Theorem A. *For $d=\text{const}\leq1$, $1\leq p, p', p''\leq\infty$, $f\in L_{\max\{p,p'\}}$, we have*

$$\tau_k(f, w;\delta)_{p,p} \textit{ is a norm in } L_p/H_{k-1}(p'\leq p); \tag{1.3}$$

$$\tau_k(f, Bw;\delta)_{p',p}=B\tau_k(f, w;\delta)_{p',p} \textit{ for } B=\text{const}>0; \tag{1.4}$$

$$\tau_k(f, w;\delta)_{p',p}\leq\tau_k(f, w;\delta)_{p'',p} \textit{ for } p'\leq p''; \tag{1.5}$$

$$\tau_k(f, w;\Delta(d))_{p',p}\leq c(k)\| wf\|_p \textit{ for } p'\leq p,\ w\in W; \tag{1.6}$$

$$\tau_k(f, w;\Delta(d))_{p',p}\leq c(k)\| w\Delta^k(d)f^{(k)}\|_p \textit{ for } f^{(k)}\in L_p,\ w\in W; \tag{1.7}$$

$$\tau_k(f, 1;d)_{p',p}\asymp\omega_k(f;d)_p \textit{ for } 1\leq p'\leq p, \tag{1.8}$$

where $\omega_k(f;d)_p$ are the usual L_p moduli of continuity.

For $f\in L_1[a, b]$ and $\sigma\neq0$ the k-th Steklov function is given by

$$f_{k,\sigma}(x)=\sigma^{-k}\int_0^\sigma\dots\int_0^\sigma f(x+u_1+\dots+u_k)du_1\dots du_k \tag{1.9}$$

and the k-th modified Steklov function ($h \neq 0$) is

$$(1.10)\qquad F_{k,h}(x)=(-1)^{k-1}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}f_{k,rhk^{-1}}(x).$$

Let us note that $f_{k,\sigma}$ and $F_{k,h}$ are defined in $[a, b-k\sigma]$ and $[a, b-kh]$ respectively ($\sigma>0$, $h>0$).

We set $x_i^{(n)}=x_i=\cos in^{-1}\pi$ for $i=0, 1, \dots, n$, $x_{-1}=x_0$, $x_{n+1}=x_n$, $I_i=[x_{i+1}, x_{i-1}]$ for $i=0, 1, \dots, n$; $h_i=(20k)^{-1}\Delta_n(x_i)$ for $i=0, 1, \dots, n$ and $H_i=\max\{h_i, h_{i-1}\}$ for $i=1, 2, \dots, n$.

Let μ be some fix function in $[0, 1]$ with the properties; $\mu(0)=0$, $\mu(1)=1$, $0<\mu(x)<1$ for $0<x<1$ and for each $s=1, 2, \dots$ $\mu^{(s)}$ exists in $[0, 1]$ and $\mu^{(s)}(0)=\mu^{(s)}(1)=0$. We set

$$\psi_0(x)=\begin{cases}0 & \text{for } x\bar{\in} I_0;\\ \mu\left(\dfrac{x-x_1}{x_0-x_1}\right) & \text{for } x\in I_0;\end{cases}\quad\text{and}\quad \psi_i(x)=\begin{cases}0 & \text{for } x\bar{\in} I_i;\\ 1-\psi_{i-1}(x) & \text{for } x\in I_i\cap I_{i-1};\\ \mu((x-x_{i+1})/(x_i-x_{i+1})) & \text{for } x\in I_i\setminus I_{i-1};\end{cases}$$

or $i=1, 2, \dots, n$. Obviously $\Sigma_{i=0}^{n}\psi_i(x)=1$ for $x\in[-1, 1]$.

2. Preliminaries. The following inequalities will be often used:

$$(2.1)\qquad (4\lambda+2)^{-1}\Delta_n(x)\leq\Delta_n(y)\leq(2\lambda+3/2)\Delta_n(x)$$

for each $x, y\in[-1, 1]$, $|x-y|\leq\lambda\Delta_n(x)$ if $\lambda>0$ and $n\geq 2\lambda$.

$$(2.2)\qquad \Delta_n(t)\leq(11/2)\Delta_n(z) \text{ for each } t, z\in I_i\cap I_{i-1}\quad (i=1, 2, \dots, n);$$

$$(2.3)\qquad |\Delta_h^k f(x)|\leq c(k)\sum_{r=0}^{k}\omega_k(f, x+rh; h)_1.$$

The inequality (2.1) follows from (2.5) in [7]. In this paper the inequality is proved under the condition $\lambda\geq 1$, but the proof remains the same one if $0<\lambda<1$. The inequality (2.2) is Lemma 1 in [2] and the inequality (2.3) follows from (3.5) in [7] with $h=s$ and $p'=1$. The inequality $\sin(\pi x/2)\geq x$ ($0\leq x\leq 1$) will be also often used.

The following three lemmas are simple embedding theorems. Their proofs are given because we need the explicit form of the dependence of constants on the parameters.

Lemma 2.1. *There is $c(k)$ such that for each $f\in W_p^k[0, 1]$ ($1\leq p\leq\infty$) $x_0\in[0, 1]$ and $j=0, 1, \dots, k-1$ the inequality*

$$|f^{(j)}(x_0)|\leq c(k)\{\|f\|_{L_p[0,1]}+\|f^{(k)}\|_{L_p[0,1]}\}$$

holds true.

Proof. It is shown in [3, p. 163, 4.4.1(6)] that there is $c(k)$ such that

$$|f^{(j)}(x_0)|\leq c(k)\{\|f\|_{L_p[0,1]}+\|R(x; x_0)\|_{(x)L_p[0,1]}\}$$

for each $x_0\in[0, 1]$, where

$$R(x; x_0)=f(x)-\sum_{j=0}^{k-1}f^{(j)}(x_0)(x-x_0)^j/j!.$$

If $f\in W_1^k$, then (see e. g. [3, p. 164, 4.4.3(1)])

$$R(x;x_0)=\frac{1}{(k-1)!}\int_{x_0}^{x}(x-u)^{k-1}f^{(k)}(u)du$$

and therefore

$$|R(x;x_0)|\leq\frac{1}{(k-1)!}\int_0^1|f^{(k)}(u)|du\leq\|f^{(k)}\|_{L_p[0,1]}.$$

This proves the lemma.

Lemma 2.2. *There is $c(k)$ such that for each $g\in W_p^k[a,b]$ $(1\leq p\leq\infty)$, $x_0\in[a,b]$ and $j=0,1,\ldots,k-1$ we have $(b-a)^j|g^{(j)}(x_0)|\leq c(k)\{\|g\|_{L_p[a,b]}+(b-a)^k\|g^{(k)}\|_{L_p[a,b]}\}(b-a)^{-1/p}$ and $(b-a)^j\|g^{(j)}\|_{L_p[a,b]}\leq c(k)\{\|g\|_{L_p[a,b]}+(b-a)^k\|g^{(k)}\|_{L_p[a,b]}\}$.*

Proof. We set $f(t)=g(a+(b-a)t)$ for $t\in[0,1]$. Then $f\in W_p^k[0,1]$ and for $r=0,1,\ldots,k$ we have

$$g^{(r)}(x)=(b-a)^{-r}f^{(r)}\left(\frac{x-a}{b-a}\right)\text{ and }\|g^{(r)}\|_{L_p[a,b]}=(b-a)^{-r}\|f^{(r)}\|_{L_p[0,1]}(b-a)^{1/p}.$$

These two equalities and Lemma 2.1 prove the first inequality in Lemma 2.2. The second inequality follows immediately from the first one.

Lemma 2.3. *Let w be a positive weight satisfying the condition*

$$(2.4)\qquad w(x)\leq Mw(t)\quad\text{for each}\quad x,t\in[a,b],\ |x-t|\leq(b-a)n^{-1}.$$

Then under the conditions of Lemma 2.2 we have

$$(b-a)^j n^{-j}\|wg^{(j)}\|_{L_p[a,b]}\leq Mc(k)\{\|wg\|_{L_p[a,b]}+(b-a)^k n^{-k}\|wg^{(k)}\|_{L_p[a,b]}\}.$$

Proof. We set $y_i=a+in^{-1}(b-a)$ for $i=0,1,\ldots,n$. Lemma 2.2 with $[a,b]=[y_i,y_{i+1}]$ $(i=0,1,\ldots,n-1)$ gives

$$(b-a)^j n^{-j}\|g^{(j)}\|_{L_p[y_i,y_{i+1}]}\leq c(k)\{\|g\|_{L_p[y_i,y_{i+1}]}+(b-a)^k n^{-k}\|g^{(k)}\|_{L_p[y_i,y_{i+1}]}\},$$

or

$$(b-a)^{pj}n^{-pj}\int_{y_i}^{y_{i+1}}|g^{(j)}(x)|^p dx\leq(c(k))^p 2^{p-1}\{\int_{y_i}^{y_{i+1}}|g(x)|^p dx$$

$$+(b-a)^{kp}n^{-kp}\int_{y_i}^{y_{i+1}}|g^{(k)}(x)|^p dx\}.$$

This inequality and (2.4) give

$$(b-a)^{pj}n^{-pj}\int_{y_i}^{y_{i+1}}|w(x)g^{(j)}(x)|^p dx\leq(b-a)^{pj}n^{-pj}\|w\|^p_{C[y_i,y_{i+1}]}\int_{y_i}^{y_{i+1}}|g^{(j)}(x)|^p dx$$

$$(2.5)\quad\leq(c(k))^p\|w\|^p_{C[y_i,y_{i+1}]}\{\int_{y_i}^{y_{i+1}}|g(x)|^p dx+(b-a)^{kp}n^{-kp}\int_{y_i}^{y_{i+1}}|g^{(k)}(x)|^p dx$$

$$\leq(c(k))^p M^p\{\int_{y_i}^{y_{i+1}}|w(x)g(x)|^p dx+(b-a)^{kp}n^{-kp}\int_{y_i}^{y_{i+1}}|w(x)g^{(k)}(x)|^p dx.$$

Taking a sum in this inequality over i $(0 \le i \le n-1)$ we obtain

$$(b-a)^{pj} n^{-pj} \int_a^b |w(x) g^{(j)}(x)|^p dx \le (c(k)M)^p \{ \int_a^b |w(x) g(x)|^p dx$$

$$+ (b-a)^{kp} n^{-kp} \int_a^b |w(x) g^{(k)}(x)|^p dx \}.$$

Using the concavity of the function $x^{1/p}$ if $p \ge 1$, from the above inequality we obtain the statement of the lemma.

Lemma 2.4. *Let* $f \in L_p[a, b]$ $(1 \le p \le \infty)$. *If* $0 < A^{-1} \le \delta_1(x)/\delta_2(x) \le 1$, *then* $\omega_k(f, x; \delta_1(x))_p \le A \omega_k(f, x; \delta_2(x))_p$.

Proof. Using (1.2) and $A \ge 1$, we have

$$\omega_k(f, x; \delta_1(x))_p = [\frac{1}{2\delta_1} \int_{-\delta_1}^{\delta_1} |\Delta_v^k f(x)|^p dv]^{1/p}$$

$$\le [\frac{1}{A^{-1} 2\delta_2} \int_{-\delta_2}^{\delta_2} |\Delta_v^k f(x)|^p dv] \le A \omega_k(f, x; \delta_2(x))_p.$$

Lemma 2.5. *If* $f \in L_p[a, b]$, $h > 0$, *then* $F_{k,h} \in W_p^k[a, b-kh]$ *and*

(2.6) $$|F_{k,h}(x) - f(x)| \le 2k \omega_k(f, x; h)_1$$

(2.7) $$|F_{k,h}^{(k)}(x)| \le c(k) h^{-k} \sum_{r=0}^{k^2} \omega_k(f, x + rhk^{-1}; h)_1 \text{ a. e.}$$

Proof. From (1.9) and (1.10) we have

$$|F_{k,h}(x) - f(x)| = |\sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} \left(\frac{k}{rh}\right)^k \int_0^{rhk^{-1}} \dots \int_0^{rhk^{-1}} f(x + t_1 + \dots + t_k) dt_1 \dots dt_k$$

$$+ (-1)^k f(x)| = |\sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} h^{-k} \int_0^h \dots \int_0^h f(x + rk^{-1}(u_1 + \dots + u_k)) du_1 \dots du_k$$

$$+ (-1)^k f(x)| = h^{-k} |\int_0^h \dots \int_0^h \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} f(x + rk^{-1}(u_1 + \dots + u_k)) du_1 \dots du_k|$$

$$\le h^{-k} \int_0^h \dots \int_0^h |\Delta_{(u_1+\dots+u_k)/k}^k f(x)| du_1 \dots du_k.$$

If we change the variables in the above integral as follows $u_1 = kt_1 - t_2 - \dots - t_k$, $u_r = t_r$ $(r = 2, 3, \dots, k)$, we obtain $|F_{k,h}(x) - f(x)| \le h^{-k} \int_0^h \dots \int_0^h |\Delta_t^k f(x)| \times k . dt_1 \dots dt_k = kh^{-1} \int_0^h |\Delta_t^k f(x)| dt \le 2k \omega_k(f, x; h)_1$.

So (2.6) is proved. As $f \in L_1[a, b]$, we have $\frac{d}{dx} \int_a^x f(t) dt = f(x)$ almost everywhere. Therefore $f_{k,\sigma}^{(k)}(x) = \sigma^{-k} \Delta_\sigma^k f(x)$ a. e. and

$$F_{kh}^{(k)}(x) = \sum_{r=1}^{k} (-1)^{1+r} \binom{k}{r} \left(\frac{k}{rh}\right)^k \Delta_{rhk^{-1}}^k f(x) \text{ a. e.}$$

This representation of $F_{k,h}^{(k)}$ shows that $F_{k,h} \in W_p^k[a, b-kh]$. Using (2.3), we have $|F_{k,h}^{(k)}(x)| \leq c(k)h^{-k}\Sigma_{r=1}^k |\Delta_{rhk^{-1}}^k f(x)| \leq c(k)h^{-k}\Sigma_{r=1}^k \Sigma_{s=0}^k \omega_k(f, x+srhk^{-1}; h)_1 \leq c(k)h^{-k} \Sigma_{r=0}^{k^2} \omega_k(f, x+rhk^{-1}; h)_1$ a. e.

Lemma 2.6. *If* $x \in I_i \cap I_{i-1}$ $(i=1, 2, \ldots, n)$, $|y| \leq 1$, $|x-y| \leq kH_i$, *then*

$$H_i \leq k^{-1}\Delta_n(y); \tag{2.8}$$

$$(230k)^{-1}\Delta_n(y) \leq \min\{h_i, h_{i-1}\}. \tag{2.9}$$

Proof. Using (2.2), we have

$$H_i = (20k)^{-1} \max\{\Delta_n(x_i), \Delta_n(x_{i-1})\} \leq (20k)^{-1} . 11/2 . \Delta_n(x) = 11\Delta_n(x)/(40k). \tag{2.10}$$

Therefore

$$|x-y| \leq k(11/40k)\Delta_n(x) = 11\Delta_n(x)/40. \tag{2.11}$$

From (2.10), (2.11) and (2.1) with $\lambda = 11/40$ $(n \geq 1 > 2\lambda)$ we obtain $H_i \leq (11/40k)\Delta_n(x) \leq (11/40k)(4(11/40)+2)\Delta_n(y) \leq k^{-1}\Delta_n(x)$. This proves (2.8). From (2.11), (2.1) with $\lambda = 11/40$ and (2.2) we have

$$\Delta_n(y) \leq (2(11/40)+3/2)\Delta_n(x) = (41/20)\Delta_n(x) \leq (41/20)(11/2) \min\{\Delta_n(x_i), \Delta_n(x_{i-1})\}$$

$$= (451/40)\, 20k \min\{h_i, h_{i-1}\} \leq 230k \min\{h_i, h_{i-1}\}.$$

This proves (2.9).

Lemma 2.7. *For each* $x \in I_i \cap I_{i-1}$, $i=1, 2, \ldots, n$, *we have* $(2/11)\Delta_n(x) \leq x_{i-1} - x_i \leq 5\Delta_n(x)$.

Proof. Using the inequality $\sin((2i-1)\pi/2n) \geq \sin(\pi/2n) \geq 1/n$, we obtain $\Delta_n(\cos((2i-1)\pi/2n)) = n^{-1}(\sin((2i-1)\pi/2n) + n^{-1}) \leq 2n^{-1}\sin((2i-1)\pi/2n)$ Therefore

$$\begin{aligned} x_{i-1} - x_i &= \cos((i-1)\pi/n) - \cos(i\pi/n) = 2\sin((2i-1)\pi/2n)\sin(\pi/n) \\ &\geq 2n^{-1}\sin((2i-1)\pi/2n) \geq \Delta_n(\cos((2i-1)\pi/2n)). \end{aligned} \tag{2.12}$$

Now (2.12) and (2.2) with $z=(2i-1)\pi/2n$ give the first inequality. Let. $x = \cos t$. Then $|t-(2i-1)\pi/2n| \leq \pi/2n$ and

$$x_{i-1} - x_i = 2\sin((2i-1)\pi/2n)\sin(\pi/2n) \leq \pi n^{-1}\sin(t + (((2i-1)\pi/2n) - t))$$

$$\leq \pi n^{-1}[\sin t . \cos(((2i-1)\pi/2n) - t) + \sin|t-(2i-1)\pi/2n| . \cos t]$$

$$\leq \pi n^{-1}(\sqrt{1-x^2} + \sin(\pi/2n)) \leq \pi n^{-1}(\sqrt{1-x^2} + (\pi/2n)) \leq 5\Delta_n(x).$$

This completes the proof.

Lemma 2.8. *If* $i=2, 3, \ldots, n$, *then* $I_{i-1} \supset [x_i, x_{i-1}+kH_i]$.

Proof. Because of $I_{i-1} = [x_i, x_{i-2}]$, we shall only prove, that $x_{i-2} - x_{i-1} - kh_j \geq 0$ for $j=i, i-1$. We have $x_{i-2} - x_{i-1} - kh_j = \cos((i-2)\pi/n) - \cos((i-1)\pi/n) - (1/20)\Delta_n(x_j) = 2\sin((2i-3)\pi/2n)\sin(\pi/2n) - (1/20)\Delta_n(x_j) \geq 2n^{-1}\sin((2i-3)\pi/2n) - (1/20)(\sin j\pi/n + n^{-1}) \geq (1/20n)(39\sin((2i-3)\pi/2n) + \sin((2i-3)\pi/2n) - \sin(j\pi/n) - n^{-1}) \geq (1/20n)(39\sin(\pi/2n) - |((2i-3)\pi/2n) - (j\pi/n)| - n^{-1}) \geq (1/20n)(39/n - 3\pi/2n - 1/n) = (76-3\pi)/40n^2 > 0$.

3. An Interpolation Theorem.

Lemma 3.1. *Let* w *be a positive weight satisfying the condition*

$$w(x) \leq Mw(t) \text{ for each } x, t \in [-1, 1], |x-t| \leq 6\Delta_n(x). \tag{3.1}$$

Then for every $f\in L_p[-1,1]$ $(1\leq p\leq\infty)$ *and for each* $k,n\in N(n\geq 6)$ *there is* $g_{k,n}\in W_p^k[-1,1/2]$ *such that*

(3.2) $\quad |g_{k,n}(x)-f(x)|\leq c(k)\omega_k(f,x;\ \Delta_n(x))_1$ *for* $x\in[-1,1/2]$ *and*

(3.3) $$\|w(\Delta_n)^k g_{k,n}^{(k)}\|_{L_p[-1,1/2]}\leq Mc(k)\,\tau_k(f,w;\Delta_n)_{1,p[-1,1]}.$$

Proof. We set

(3.4) $$g_{k,n}(x)=\sum_{i=0}^{n}F_{k,h_i}(x)\psi_i(x).$$

Because of $kh_i=\Delta_n(x_i)/20\leq 1/10$, we have $x+kh_i\in[-1,3/5]\subset[-1,1]$ if $x\in[-1,1/2]$ and hence $g_{k,n}$ is defined in $[-1,1/2]$. Lemma 2.5 and the definition of ψ_i show that $g_{k,n}\in W_p^k[-1,1/2]$. (3.4) and (1.11) give

(3.5) $$|g_{k,n}(x)-f(x)|=|\sum_{i=0}^{n}F_{k,h_i}(x)\psi_i(x)-f(x)\sum_{i=0}^{n}\psi_i(x)|\leq\sum_{i=0}^{n}|F_{k,h_i}(x)-f(x)|\psi_i(x).$$

Let i be such that $x\in I_i\cap I_{i-1}$. The definition of ψ_i, (3.5) and (2.6) give

(3.6) $$|g_{k,n}(x)-f(x)|\leq|F_{k,h_i}(x)-f(x)|+|F_{k,h_{i-1}}(x)-f(x)|$$
$$\leq 2k[\omega_k(f,x;h_i)_1+\omega_k(f,x;h_{i-1})_1].$$

Now (3.6), Lemma 2.6 and Lemma 2.4 prove (3.2).

To prove (3.3) we consider $x\in I_i\cap I_{i-1}$. From (3.4) and the definition of ψ_i, the equality

(3.7) $$g_{k,n}(x)=F_{k,h_i}(x)+\Phi(x)\psi_{i-1}(x)$$

follows, where $\Phi(x)=F_{k,h_{i-1}}(x)-F_{k,h_i}(x)$. Using (2.6), Lemma 2.4 and Lemma 2.6 with $x=y$, we obtain

(3.8) $$\|\Phi\|_{L_p[x_i,x_{i-1}]}=\|F_{k,h_{i-1}}-f+f-F_{k,h_i}\|_{L_p[x_i,x_{i-1}]}\leq\|F_{k,h_{i-1}}-f\|_{L_p[x_i,x_{i-1}]}$$
$$+\|F_{k,h_i}-f\|_{L_p[x_i,x_{i-1}]}\leq 2k\{\|\omega_k(f,x;h_{i-1})_1\|_{L_p[x_i,x_{i-1}]}$$
$$+\|\omega_k(f,x;h_i)_1\|_{L_p[x_i,x_{i-1}]}\}\leq c(k)\|\omega_k(f,x;\Delta_n(x))_1\|_{L_p[x_i,x_{i-1}]}.$$

Using (2.7), Lemma 2.6 and Lemma 2.7, we have

(3.9) $$(x_{i-1}-x_i)^k\|\Phi^{(k)}\|_{L_p[x_i,x_{i-1}]}\leq(x_{i-1}-x_k)^k\{\|F_{k,h_{i-1}}^{(k)}\|_{L_p[x_i,x_{i-1}]}$$
$$+\|F_{k,h_i}^{(k)}\|_{L_p[x_i,x_{i-1}]}\}\leq(x_{i-1}-x_i)^k c(k)\sum_{r=0}^{k^2}\{h_{i-1}^{-k}\|\omega_k(f,x+rh_{i-1}/k;h_{i-1})_1\|_{L_p[x_i,x_{i-1}]}$$
$$+h_i^{-k}\|\omega_k(f,x+rh_i/k;h_i)_1\|_{L_p[x_i,x_{i-1}]}\}\leq c(k)\sum_{i=0}^{k^2}\{\|\omega_k(f,x+rh_{i-1}/k;$$
$$\Delta_n(x+rh_{i-1}/k))_1\|_{L_p[x_i,x_{i-1}]}+\|\omega_k(f,x+rh_i/k;\Delta_n(x+rh_i/k))_1\|_{L_p[x_i,x_{i-1}]}\}$$
$$\leq c(k)\sum_{i=0}^{k^2}\{\|\omega_k(f,x;\Delta_n(x))_1\|_{L_p[x_i+rh_{i-1}/k,x_{i-1}+rh_{i-1}/k]}+\|\omega_k(f,x;$$
$$\Delta_n(x))_1\|_{L_p[x_i+rh_i/k,x_{i-1}+rh_i/k]}\}\leq c(k)\|\omega_k(f,x;\Delta_n(x))_1\|_{L_p[x_i,x_{i-1}+kH_i]}.$$

Now using (3.7), the definition of ψ_{i-1}, Lemma 2.2, (3.8), (3.9) and (2.7), we get that the following inequality is satisfied almost everywhere in $[x_i, x_{i-1}]$:

$$|g_{k,n}^{(k)}(x)|\leq |F_{k,h_i}^{(k)}(x)|+\sum_{j=0}^{k}\binom{k}{j}|\Phi^{(j)}(x)||\psi_{i-1}^{(k-j)}(x)|$$

$$\leq |F_{k,h_i}^{(k)}(x)|+\sum_{j=0}^{k}\binom{k}{j}|\Phi^{(j)}(x)|(x_{i-1}-x_i)^{j-k}\|\lambda^{(k-j)}\|_{C[0,1]}\leq |F_{k,h_i}^{(k)}(x)|$$

$$+|\Phi^{(k)}(x)|+c(k)(x_{i-1}-x_i)^{-k}\sum_{j=0}^{k-1}(x_{i-1}-x_i)^j|\Phi^{(j)}(x)|\leq 2|F_{k,h_i}^{(k)}(x)|$$

$$+|F_{k,h_{i-1}}^{(k)}(x)|+c(k)(x_{i-1}-x_i)^{-k-1/p}\{\|\Phi\|_{L_p[x_i, x_{i-1}]}+(x_{i-1}-x_i)^k\|\Phi^{(k)}\|_{L_p[x_i,x_{i-1}]}\}$$

$$\leq c(k)(x_{i-1}-x_i)^{-k}\{\sum_{r=0}^{k^2}[\omega_k(f, x+rh_i/k; h_i)_1+\omega_k(f, x+rh_{i-1}/k; h_{i-1})_1]$$

$$+(x_{i-1}-x_i)^{-1/p}\|\omega_k(f, x; \Delta_n(x))_1\|_{L_p[x_i,x_{i-1}+kH_i]}\}.$$

The above inequality and Lemma 2.7 give

$$\|w(\Delta_n)^k g_{k,n}^{(k)}\|_{L_p[-1,1/2]}\leq [\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}}|w(x)(\Delta_n(x))^k g_{k,n}^{(k)}(x)|^p dx]^{1/p} \tag{3.10}$$

$$\leq c(k)[\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}}w^p(x)\{\sum_{r=0}^{k^2}[\omega_k(f, x+rh_i/k; h_i)_1+\omega_k(f, x+rh_{i-1}/k; h_{i-1})_1]$$

$$+(x_{i-1}-x_i)^{-1/p}\|\omega_k(f, x; \Delta_n(x))_1\|_{L_p[x_i, x_{i-1}+kH_i]}\}^p dx]^{1/p}$$

$$\leq c(k)\sum_{r=0}^{k^2}\{[\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}}w^p(x)\,\omega_k^p(f, x+rh_{i-1}/k; h_{i-1})_1\,dx]^{1/p}$$

$$+[\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}}w^p(x)\omega_k^p(f, x+rh_i/k; h_i)_1 dx]^{1/p}\}$$

$$+c(k)[\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}}\frac{w^p(x)}{x_{i-1}-x_i}\int_{x_i}^{x_{i-1}+kH_i}\omega_k^p(f, u; \Delta_n(u))_1 du dx]^{1/p}.$$

Let $s=i$ or $s=i-1$, $0\leq r\leq k^2$ and $x_i\leq x\leq x_{i-1}$. Lemma 2.6 and Lemma 2.4 give

$$\omega_k(f, x+rh_s/k; h_s)_1\leq c(k)\omega_k(f, x+rh_s/k; \Delta_n(x+rh_s/k))_1. \tag{3.11}$$

(2.2) gives $rh_s/k\leq kh_s=\Delta_n(x_s)/20\leq 11\Delta_n(x)/40$. From the above inequality and (3.1) we have

$$w(x)\leq Mw(x+rh_s/k). \tag{3.12}$$

Applying Lemma 2.7 and Lemma 2.6, for $u\in[x_i, x_{i-1}+kH_i]$ we have

$$|x-u|\leq x_{i-1}-x_i+kH_i\leq 5\Delta_n(x)+\Delta_n(x)=6\Delta_n(x). \tag{3.13}$$

(3.11), (3.12) and Lemma 2.8 give

$$(3.14)\qquad [\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}} w^p(x)\omega_k^p(f, x+rh_s/k; h_s)_1 dx]^{1/p}$$

$$\leq Mc(k)[\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}} w^p(x+rh_s/k)\omega_k^p(f, x+rh_s/k; \Delta_n(x+rh_s/k))_1 dx]^{1/p}$$

$$\leq Mc(k)[\sum_{i=[n/3]}^{n}\int_{x_i+rh_s/k}^{x_{i-1}+rh_s/k} w^p(x)\omega_k^p(f, x; \Delta_n(x))_1 dx]^{1/p}$$

$$\leq Mc(k)[\sum_{i=[n/3]}^{n}\int_{I_{i-1}} w^p(x)\omega_k^p(f, x; \Delta_n(x))_1 dx]^{1/p}$$

$$\leq Mc(k)[2\sum_{i=1}^{n}\int_{x_i}^{x_{i-1}} w^p(x)\omega_k^p(f, x; \Delta_n(x))_1 dx]^{1/p} = Mc(k)\tau_1(f, w; \Delta_n)_{1,p[-1,1}.$$

From (3.13), (3.1) and Lemma 2.8 we get

$$(3.15)\qquad [\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}} (x_{i-1}-x_i)^{-1}w^p(x)\int_{x_i}^{x_{i-1}+kH_i}\omega_k^p(f, u; \Delta_n(u))_1 du dx]^{1/p}$$

$$\leq M[\sum_{i=[n/3]}^{n}\int_{x_i}^{x_{i-1}} (x_{i-1}-x_i)^{-1}\int_{x_i}^{x_{i-1}+kH_i} w^p(u)\omega_k^p(f, u; \Delta_n(u))_1 du dx]^{1/p}$$

$$= M[\sum_{i=[n/3]}^{n}\int_{I_{i-1}} w^p(u)\omega_k^p(f, u; \Delta_n(u))_1 du]^{1/p} \leq 2M\tau_1(f, w; \Delta_n)_{1,p[-1,1]}.$$

Finally (3.10), (3.14) and (3.15) prove (3.3)

Theorem 3.1. *Let w be a positive weight satisfying* (3.1). *Then for every $k, n\in N$, $n\geq 6$, and for each $f\in L_p[-1,1]$ $(1\leq p\leq\infty)$ there is $G_{k,n}\in W_p^k[-1,1]$, such that*

$$(3.16)\qquad |G_{k,n}(x)-f(x)|\leq c(k)\omega_k(f, x; \Delta_n(x))_1$$

for $x\in[-1,1]$,

$$(3.17)\qquad \|w(\Delta_n)^k G_{k,n}^{(k)}\|_p \leq M^2 c(k)\tau_k(f, w; \Delta_n)_{1,p}.$$

Proof. We apply Lemma 3.1 for the functions $f(x)$ and $\bar{f}(x)=f(-x)$ and we get a function $g_{k,n}\in W_p^k[-1,1/2]$ satisfying the conditions (3.2) and (3.3) and a function $\bar{g}_{k,n}\in W_p^k[-1/2,1]$, for which

$$(3.18)\qquad |\bar{g}_{k,n}(x)-f(x)|\leq c(k)\omega_k(f, x; \Delta_n(x))_1 \text{ for } x\in[-1/2,1] \text{ and}$$

$$(3.19)\qquad \|w(\Delta_n)^k \bar{g}_{k,n}^{(k)}\|_{L_p[-1/2,1]} \leq Mc(k)\tau_k(f, w; \Delta_n)_{1,p[-1,1]}.$$

We set

$$(3.20)\qquad G_{k,n}(x)=\begin{cases} g_{k,n}(x) & \text{for } x\in[-1,-1/2];\\ \mu(1/2+x)[\bar{g}_{k,n}(x)-g_{k,n}(x)]+g_{k,n}(x) & \text{for } x\in[-1/2,1/2];\\ \bar{g}_{k,n}(x) & \text{for } x\in[1/2,1].\end{cases}$$

Therefore $G_{k,n}\in W_p^{*k}[-1,1]$. Now (3.2), (3.18) and (3.20) prove (3.16). If $|x|\le 1/2$ we have

$$\sqrt{3}/(2n)\le\Delta_n(x)\le 2/n. \tag{3.21}$$

(3.21) and (3.1) show that w satisfies (2.4) with $a=-1/2$, $b=1/2$ with the same constant M in (3.1). Finally, applying (3.20), (3.2), (3.3), (3.18), (3.19), (3.21) and Lemma 2.3, we get

$$\|w(\Delta_n)^k G_{k,n}^{(k)}\|_{L_p[-1,1]}\le\|w(\Delta_n)^k g_{k,n}^{(k)}\|_{L_p[-1,1/2]}+\|w(\Delta_n)^k \bar{g}_{k,n}^{(k)}\|_{L_p[1/2,1]}$$

$$+\|w(x)\Delta_n^k(x)[\mu(1/2+x)(\bar{g}_{k,n}(x)-g_{k,n}(x))]^{(k)}\|_{L_p[-1/2,1/2]}$$

$$\le c(k)M\tau_k(f,w;\Delta_n)_{1,p}+c(k)n^{-k}\sum_{r=0}^{k}\|w(x)\mu^{(k-r)}(1/2+x)[\bar{g}_{k,n}(x)-g_{k,n}(x)]^{(r)}\|_{L_p[-1/2,1/2]}$$

$$\le c(k)M\tau_k(f,w;\Delta_n)_{1,p}+c(k)n^{-k}\sum_{r=0}^{k}\|w(\bar{g}_{k,n}-g_{k,n})^{(r)}\|_{L_p[-1/2,1/2]}$$

$$\le c(k)M\tau_k(f,w;\Delta_n)_{1,p}+Mc(k)\sum_{r=0}^{k}n^{r-k}\{\|w(\bar{g}_{k,n}-g_{k,n})\|_{L_p[-1/2,1/2]}$$

$$+n^{-k}\|w(\bar{g}_{k,n}-g_{k,n})^{(k)}\|_{L_p[-1/2,1/2]}\}\le c(k)M\tau_k(f,w;\Delta_n)_{1,p}$$

$$+Mc(k)\{\|w(\bar{g}_{k,n}-f)\|_{L_p[-1/2,1/2]}+\|w(g_{k,n}-f)\|_{L_p[-1/2,1/2]}+\|w(\Delta_n)^k g_{k,n}^{(k)}\|_{L_p[-1/2,1/2]}$$

$$+\|w(\Delta_n)^k\bar{g}_{k,n}^{(k)}\|_{L_p[-1/2,1/2]}\}\le M^2c(k)\tau_k(f,w;\Delta_n)_{1,p[-1,1]}.$$

This completes the proof of the theorem.

4. A Characterization of the Best Algebraic Approximation. The following two theorems are announced in [6] and at the International Conference on "Functions, Series, Operators", August 21-29, 1980, Budapest. Their proofs will appear in [2].

Theorem B. *Let K be a non-negative integer and $w\in W(s)$ for some $s>0$. Then for each f such that $f^{(k)}\in L_p[-1,1]$ $(1\le p\le\infty)$ we have*

$$E_{n+k}(w;f)_p\le c(k,s)\,\tau_1(f^{(k)},w(\Delta_n)^k;\Delta_n)_{1,p}.$$

Theorem C. *For $k\in\mathbf{N}$, $w\in W_k$, $1\le p\le\infty$, $f\in L_p[-1,1]$, we have*

$$\tau_k(f,w;\Delta_n)_{p,p}\le c(k)n^{-k}\sum_{s=0}^{n}(s+1)^{k-1}E_s(w;f)_p.$$

Now we shall prove the following Steckin's type theorem.

Theorem 4.1. *If $k\in\mathbf{N}$, $w\in W(s)$ for some $s>0$, $f\in L_p[-1,1]$ $(1\le p\le\infty)$, then*

$$E_{n+k}(w;f)_p\le c(s,k)\,\tau_k(f,w;\Delta_n)_{1,p}.$$

Proof. We consider $G_{k,n}$ in Theorem 3.1 for our function f. Using Theorem 3.1, Theorem B and (1.7), we obtain

$$E_{n+k}(w;f)_p \leqq \| w(f-G_{k,n})\|_p + E_{n+k}(w;G_{k,n})_p$$

$$\leqq c(k)\, \| w(x)\,\omega_k(f,x;\Delta_n(x))_1\|_p + c(s,k)\,\tau_1(G_{k,n}^{(k-1)},(\Delta_n)^{k-1}w;\Delta_n)_{1,p}$$

$$\leqq c(k)\,\tau_k(f,w;\Delta_n)_{1,p} + c(s,k)\,\| w(\Delta_n)^k G_{k,n}^{(k)}\|_p \leqq c(k,s)\,\tau_k(f,w;\Delta_n)_{1,p}.$$

This proves the theorems.

Now Theorem 4.1, Theorem C and (1.5) give

Corollary 4.1. *For* $k\in N$, $w\in W(s)\cap W_k$ *for some* $s>0$, $f\in L_p[-1,1]$, $(1\leqq p\leqq\infty)$, $0<\alpha<k$, $\gamma\geqq 0$, *we have*

$$E_n(w;f)_p = O(n^{-\alpha}) \Leftrightarrow \tau_k(f,w;\Delta_n)_{1,p} = O(n^{-\alpha})$$

and in particular

$$E_n(w_{n,\gamma};f)_p = O(n^{-\alpha}) \Leftrightarrow \tau_k(f,w_{n,\gamma};\Delta_n)_{1,p} = O(n^{-\alpha}),$$

$$E_n(f)_p = O(n^{-\alpha}) \Leftrightarrow \tau_k(f,1;\Delta_n)_{1,p} = O(n^{-\alpha}).$$

REFERENCES

1. А. С. Джафаров. Осредненные модули непрерывности и некоторые связи их и наилучшими приближениями. *Доклады АН СССР,* **236**, 1977, № 2, 288-291.
2. K. G. Ivanov. On a new characteristic of functions. II. Direct and converse theorems for the best algebraic approximation in $C[-1,1]$ and $L_p[-1,1]$. *Pliska,* **5**, 1983, 151-163.
3. С. М. Никольский. Приближение функций многих переменных и теоремы вложения. Москва, 1977.
4. М. К. Потапов. О приближении многочленами Якоби. *Вестн. Моск. ун-та, Сер. мат. мех.,* 1977, № 5, 70-82.
5. А. Л. Фуксман. Структурная характеристика функций, у которых $E_n(f;-1,1) \leqq Mn^{-(k+\alpha)}$. *Успехи мат. наук,* **20**, 1965, № 4, 187-190.
6. K. G. Ivanov. Direct and converse theorems for the best algebraic approximation in $C[-1,1]$ and $L_p[-1,1]$. *C. R. Acad. bulg. Sci.,* **33**, 1980, No 10, 1309-1312.
7. K. G. Ivanov. On a new characteristic of functions, 1. *Serdika,* **8**, 1982, 262-279.
8. R. L. Stens, M. Wehrens. Legendre transform methods and best algebraic approximation. *Ann. Soc. Math. Pol., Ser. I,* 1979, 351-380.

Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373 Bulgaria

Received on June 26, 1981