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LOCAL POLYNOMIAL APPROXIMATION AND LIPSCHITZ FUNCTIONS ON CLOSED SETS

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Summary. By means of local polynomial approximation we characterize the spaces $\Lambda_\alpha(F)$, $\alpha>0$, of Lipschitz continuous functions, corresponding to higher order differences, on an arbitrary closed subset F of R^n, and the Besov spaces $B_\alpha^{p,q}(F)$, $\alpha>0$, $1\leq p$, $q\leq\infty$, of functions defined on a closed subset F of R^n of a general type. We also give a unified introduction to these spaces $\Lambda_\alpha(F)$ and $B_\alpha^{p,q}(F)$. They are restrictions to F of certain Lipschitz or Besov spaces in R^n and were introduced and studied in [3—6] for a general F.

Introduction. Whitney's extension theorem [9, Ch. VI] characterizes by means of smoothness conditions the space Lip(α, F), $\alpha>0$, which gives the restriction to closed subsets F of R^n of the space Lip (α, R^n); see Section 1.1 for the definitions. Lip(α, R^n) is defined by means of first differences. If we replace these by second differences, we get (Section 1.1) the Lipschitz spaces $\Lambda_\alpha(R^n)$, which is different from Lip(α, R^n), if α is an integer. The trace space of $\Lambda_\alpha(R^n)$ was determined in [5] (see also [10] and Theorem 1.1 below) and consists of the space $\Lambda_\alpha(F)$ in Definition 1.2. The purpose here is to analyse these spaces $\Lambda_\alpha(F)$ in more detail. We characterize them by means of local polynomial approximation in Theorem 1.2 and in Theorem 1.3 we consider some closely related spaces. If Markov's inequality for polynomials holds in a certain sense on F (Definition 1.3), then the conditions are simplified (Theorem 1.4). In Section 2 we briefly state analogous results for the Besov spaces $B_\alpha^{p,q}(F)$.

1. **The Spaces** $\Lambda_\alpha(F)$. We assume throughout that F is a non-empty closed subset of the n-dimensional Euclidean space R^n with points $x=(x_1, \ldots, x_n)$ and distance $|x|$; k is a non-negative integer and $k<\alpha\leq k+1$; if $j=(j_1, \ldots, j_n)$ is a multi-integer with length $|j|=j_1+\cdots+j_n$, then $D^j f$ is the corresponding partial derivative of order $|j|$, $x^j=x_1^{j_1}\ldots x_n^{j_n}$, and $j!=j_1!, \ldots, j_n!$; $\Delta_h^m f(x)$ denotes the difference of order m with step h, i. e. $\Delta_h^1 f(x)=f(x+h)-f(x)$ and, for $m>1$, $\Delta_h^m f(x)=\Delta_h^1(\Delta_h^{m-1}f)(x)$; $M, c, c_1, \ldots$ denote constants, usually different each time they appear.

1.1. The spaces Lip(α, F) and $\Lambda_\alpha(R^n)$. For a family of functions $\{f^{(j)}\}$, $|j|\leq k$, j a multi-integer, defined on F, we introduce the formal Taylor expansion remainders R_j, $|j|\leq k$, of $\{f^{(j)}\}$ by

$$(1.1)\qquad R_j(x,y)=f^{(j)}(x)-\sum_{|j+l|\leq k}\frac{f^{(j+l)}(y)}{l!}(x-y)^l.$$

Definition 1.1. $\mathrm{Lip}(\alpha, F)$, $\alpha>0$, *consists of those families* $\{f^{(j)}\}$, $|j|\leq k$, *for which* $|f^{(j)}(x)|\leq M$ *and* $|R_j(x,y)|\leq M|x-y|^{\alpha-|j|}$, *for all* $x, y\in F$, $|j|\leq k$. *The norm of* $\{f^{(j)}\}$ *in* $\mathrm{Lip}(\alpha, F)$ *is the infimum of the constants* M.

If $F=R^n$, the situation is simplified: $\{f^{(j)}\}\in \mathrm{Lip}(\alpha, R^n)$ if and only if $f^{(j)}=D^jf$, where $f=f^{(0)}$, and D^jf, $|j|\leq k$, are bounded continuous functions on R^n such that D^jf, for $|j|=k$, satisfy $|\Delta_h^1D^jf(x)|\leq M|h|^{\alpha-k}$. In this case we speak of functions $f\in\mathrm{Lip}(\alpha, R^n)$ instead of the family $\{f^{(j)}\}\in\mathrm{Lip}(\alpha, R^n)$, since $f^{(j)}=D^jf$ are uniquely determined by f.

The space $\Lambda_\alpha(R^n)$ is defined analogously to the latter definition of Lip (α, R^n) but with higher order differences instead of Δ_h^1. This means [9, Ch. VI, 2.3] that $\Lambda_\alpha(R^n)=\mathrm{Lip}(\alpha, R^n)$, if $k<\alpha<k+1$ and that $\Lambda_{k+1}(R^n)$ is obtained by changing Δ_h^1 to Δ_h^2 in the definition of $\mathrm{Lip}(k+1, R^n)$.

1.2. Definition of $\Lambda_\alpha(F)$. For any closed set F we now define $\Lambda_\alpha(F)$. Let $f^{(j)}, |j|\leq k$, be functions defined on F.

Definition 1.2. *Let* $\gamma>\alpha>0$, $k<\alpha\leq k+1$, *and* $m<\gamma<m+1$, *where* k *and* m *are integers. Then* $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$ *if, for* $\nu=1, 2, \ldots$, *there exists* $\{f_\nu^{(j)}\}_{|j|\leq m}\in\mathrm{Lip}(\gamma, F)$ *such that, for* $x\in F$ *and* $\nu=1, 2, \ldots$,

a) $|f^{(j)}(x)-f_\nu^{(j)}(x)|\leq M2^{-\nu(\alpha-|j|)}$, $|j|\leq k$,

b) $|f_{\nu+1}^{(j)}(x)-f_\nu^{(j)}(x)|\leq M2^{-\nu(\alpha-|j|)}$, $k+1\leq|j|\leq m$, *if* $m>k$,

and

c) $\|\{f_\nu^{(j)}\}_{|j|\leq m}\|_{\mathrm{Lip}(\gamma, F)}\leq M2^{-\nu(\alpha-\gamma)}$.

The norm of $\{f^{(j)}\}$ *in* $\Lambda_\alpha(F)$ *is the infimum of the constants* M *for which* a) — c) *hold for some* $\{f_\nu^{(j)}\}$, $|j|\leq m$, $\nu=1, 2, \ldots$.

It will follow from Proposition 1.1 that the space $\Lambda_\alpha(F)$ does not, in fact, depend on what γ we choose. Note that $m\geq k$ and that the condition b) disappears if $m=k$, which is the case with a natural choice of γ if $k<\alpha<k+1$; however, if $\alpha=k+1$ we cannot get rid of b); compare the example in [5, Section 2.1]. It follows from the definition that the functions $f^{(j)}$ in the family $\{f^{(j)}\}\in\Lambda_\alpha(F)$ are continuous functions on F, since $f_\nu^{(j)}$ are continuous functions on F by Definition 1.1. Definition 1.2 is justified by the following three propositions and the trace Theorem 1.1; in particular Proposition 1.1 which is seemingly weaker than Definition 1.2, shows that different γ: *s* in Definition 1.2 give the same space $\Lambda_\alpha(F)$ with equivalent norms. The propositions and the theorem follow easily from the ideas introduced in [5, 6, 10]. The detailed proofs are given in [2, Section 4]. Let $[\alpha]$ denote the integer part of α and let $R_{\nu j}$, $|j|\leq[\alpha]$ be the formal Taylor expansion remainders of a family $\{f_\nu^{(j)}\}$, $|j|\leq[\alpha]$ (defined by (1.1) with $f^{(j)}$ changed to $f_\nu^{(j)}$, k to $[\alpha]$, and R_j to $R_{\nu j}$).

Proposition 1.1. *Let* a *be a positive number. Then* $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$ *if and only if there exist families* $\{f_\nu^{(j)}\}, |j|\leq[\alpha], \nu=1, 2, \ldots$, *of functions defined on* F *such that for* $x, y\in F$, $\nu=1, 2, \ldots$,

$$(1.2)\qquad |f^{(j)}(x)-f_\nu^{(j)}(x)|\leq M2^{-\nu(\alpha-|j|)},\ \textit{for}\ |j|\leq k,$$

$$(1.3)\qquad |f_{\nu+1}^{(j)}(x)-f_\nu^{(j)}(x)|\leq M,\ \textit{for}\ |j|=k+1,\ \textit{if}\ \alpha=k+1,$$

$$(1.4)\qquad |R_{\nu j}(x,y)|\leq M2^{-\nu(\alpha-|j|)},\ \textit{for}\ |j|\leq[\alpha],\ |x-y|\leq a2^{-\nu},$$

(1.5) $$\text{and } |f_1^{(j)}(x)| \leq M, \text{ for } |j| \leq [\alpha].$$

The norm of $\{f^{(j)}\}$ *in* $\Lambda_\alpha(F)$ *is equivalent to the infimum of all* M *such that* (1.2)—(1.5) *are satisfied for some* $\{f_\nu^{(j)}\}$; *in particular, different* a:*s give equivalent constants* M.

The alternative definition of $\Lambda_\alpha(F)$ given by Proposition 1.1 was used in [5], when $\alpha = k+1$. The fact that different a:*s* give equivalent constants M is easily established; see [5, Remark 1.2], where this is proved for $\alpha = 1$.

Proposition 1.2. $\Lambda_\alpha(F) = \mathrm{Lip}(\alpha, F)$ *with equivalent norms, if* $\alpha \neq k+1$.

The next proposition shows that when $F = R^n$, Definition 1.2 gives the same space $\Lambda_\alpha(R^n)$ as the classical definition in Section 1.1.

Proposition 1.3. *Assume that* $F = R^n$. *If* $\{f^{(j)}\}_{|j| \leq k} \in \Lambda_\alpha(F)$ *as defined by Definition* 1.2, *then* $f^{(j)} = D^j f$, *for* $|j| \leq k$, *where* $f = f^{(0)}$. *Furthermore,* $\{D^j f\}_{|j| \leq k} \in \Lambda_\alpha(F)$ *as defined by Definition* 1.2 *if and only if* $f \in \Lambda_\alpha(R^n)$ *as defined in Section* 1.1.

For all $f \in \Lambda_\alpha(R^n)$ we can form $(D^j f)|F$, for $|j| \leq k$, i. e. the pointwise restriction to F to $D^j f$. By Proposition 1.3 and Definition 1.2 we then clearly get elements in $\Lambda_\alpha(F)$. The content of Theorem 1.1 is that the families of functions, which we obtain in this way, give the whole space $\Lambda_\alpha(F)$.

Theorem 1.1. $\Lambda_\alpha(F) = \{\{(D^j f)|F\}_{|j| \leq k} : f \in \Lambda_\alpha(R^n)\}$. *More exactly, if* $\{f^{(j)}\}_{|j| \leq k} \in \Lambda_\alpha(F)$, *then* $f = f^{(0)}$ *may be extended to* R^n *to a function* $Ef \in \Lambda_\alpha(R^n)$ *such that* $D^j(Ef) = f^{(j)}$ *on* F *for* $|j| \leq k$, *and* $\|Ef\|_{\Lambda_\alpha(R^n)} \leq c\|\{f^{(j)}\}\|_{\Lambda_\alpha(F)}$, *where the constant* c *depends only on* n *and* α.

1.3. Characterization of $\Lambda_\alpha(F)$ by means of local polynomial approximation. We consider cubes Q and Q' with sides parallel to the axes.

Theorem 1.2. *Let* b *be a number larger than* 1. *Then a family* $\{f^{(j)}\}$, $|j| \leq k$, *of functions defined on* F *belongs to* $\Lambda_\alpha(F)$, $k < \alpha \leq k+1$, *if and only if the following condition holds: For every cube* Q *in* R^n, $Q \cap F \neq \emptyset$, *with side length* $\delta > 0$ *there exists a polynomial* P_Q *of degree* $\leq [\alpha]$ *such that* (1.6)—(1.8) *hold:*

(1.6) $$|f^{(j)}(x) - D^j P_Q(x)| \leq c\delta^{\alpha - |j|}, \text{ for } |j| \leq k, \ x \in Q \cap F.$$

If Q', $Q' \cap F \neq \emptyset$, $Q' \subset Q$, *is a cube with side of length* δ' *such that* $\delta \leq b\delta'$, *then*

(1.7) $$|D^j P_Q(x) - D^j P_{Q'}(x)| \leq c, \text{ for } |j| = k+1,$$

(1.8) $$|P_Q(x)| \leq c, \text{ for } x \in Q, \text{ if } \delta = 1.$$

The norm of $\{f^{(j)}\} \in \Lambda_\alpha(F)$ *is equivalent to the infimum of the constants* c *in* (1.6)—(1.8); *in particular, different* b:*s give equivalent* c:*s*.

Here, of course, $P_{Q'}$ denotes the polynomial of degree $\leq [\alpha]$ associated to Q'. Note that the left member of (1.7) is a constant and that this constant is zero, if $\alpha \neq k+1$, i. e. the condition (1.7) disappears if $\alpha \neq k+1$. It is easy to see that the theorem remains true if we in (1.6)—(1.8) work with the seemingly weaker condition that Q and Q' shall have center at points in F. Also, we could throughout assume $\delta \leq 1$ and we could, like in Definition 1.2, replace δ by a discrete parameter ν, $\delta \approx 2^{-\nu}$, and use polynomials of higher degree.

In the theorem the condition (1.7) may be replaced by: *If* Q' *is a cube with side of length* $\delta' > 0$ *such that* $Q' \cap F \neq \emptyset$, *then*

(1.9) $$|P_Q(x)-P_{Q'}(x)|\leq c\,(\max(\delta,\delta'))^\alpha,\ \textit{for}\ x\in Q\cap Q'.$$

In fact, for $\alpha=k+1$ the condition (1.7) follows from (1.9), with an equivalent constant c, by means of repeated application of Markov's inequality (see for instance Section 1.5). Conversely, if $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$, we may, by Theorem 1.1, assume that $f=f^{(0)}\in\Lambda_\alpha(R^n)$. If we assume that Theorem 1.2 has already been proved, this means that (1.6) holds on Q and analogously with Q replaced by Q', and hence (1.9) holds by the triangle inequality. Compare [5, Remark 4.5].

Proof of Theorem 1.2. A) Assume that $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$. We choose $\{f_\nu^{(j)}\}$, $|j|\leq[\alpha]$, $\nu=1,2,\ldots$, satisfying the conditions (1.2)—(1.5) in Proposition 1.1. For every cube Q, $Q\cap F\neq\emptyset$, with side of length $\delta>0$ we choose a point $y=y_Q\in Q\cap F$ and, if $\delta\leq 1$, $\nu=\nu_\delta$ such that $\delta/2\leq 2^{-\nu}<\delta$. We define P_Q by

$$P_Q(x)=\sum_{|l|\leq[\alpha]}\frac{f_\nu^{(l)}(y)}{l\,!}(x-y)^l,\ \text{if}\ \delta\leq 1,\ \text{and}$$

$P_Q(x)=0$, if $\delta>1$. We shall prove that (1.6)—(1.8) are satisfied. For $x\in Q\cap F$, $|j|\leq k$, and $\delta\leq 1$ we have by the definition of P_Q above and of $R_{\nu j}$ in Proposition 1.1,

$$|f^{(j)}(x)-D^jP_Q(x)|\leq|f^{(j)}(x)-f_\nu^{(j)}(x)|+|R_{\nu j}(x,y)|,$$

and, by (1.2) and (1.4), this is $\leq M2^{-\nu(\alpha-|j|)}\leq M\delta^{\alpha-|j|}$, which proves (1.6) for $\delta\leq 1$. If $\delta>1$, (1.6) follows from the facts that $P_Q=0$ and, by (1.2) and (1.5), $|f^{(j)}(x)|\leq M$. When $\delta=1$, $\nu=\nu_\delta=1$, and (1.8) follows from (1.5). Finally, we prove (1.7). Assume that $\alpha=k+1$ and let Q', $Q'\subset Q$, $Q'\cap F\neq\emptyset$, be a cube with side of length δ', $\delta\leq b\delta'$, with a point $y_{Q'}\in Q'\cap F$ and, if $\delta'\leq 1$, let $\mu=\mu_{\delta'}$ satisfy $\delta'/2\leq 2^{-\mu}<\delta'$. Then $\mu\geq\nu$, and, for $|j|=k+1$, $\delta\leq 1$, we get by (1.4) and (1.3),

$$|D^jP_Q(x)-D^jP_{Q'}(x)|=|f_\nu^{(j)}(y_Q)-f_\mu^{(j)}(y_{Q'})|$$
$$\leq|f_\nu^{(j)}(y_Q)-f_\nu^{(j)}(y_{Q'})|+|f_\nu^{(j)}(y_{Q'})-f_\mu^{(j)}(y_{Q'})|\leq M+M(\mu-\nu)\leq c,$$

which is (1.7) for $\delta\leq 1$; the case $\delta>1$, $\delta'\leq 1$ follows for instance by (1.2) and (1.5). This proves one half of Theorem 1.2.

B) Now let $\{f^{(j)}\}$, $|j|\leq k$, be a family of functions defined on F such that there exist polynomials P_Q, satisfying (1.6)—(1.8). We shall prove that $\{f^{(j)}\}\in\Lambda_\alpha(F)$. Divide, for $\nu=1,2,\ldots,R^n$ into a net $\pi_\nu=\{Q_{\nu i}\}_{i=1}^\infty$ of disjoint half-open cubes $Q_{\nu i}$ with sides of length $2^{-\nu+1}$ parallel to the axes such that we obtain the net $\pi_{\nu+1}$ by dividing each cube in π_ν into 2^n equal cubes. Define $\{f_\nu^{(j)}\}$, $|j|\leq[\alpha]$, by

$$f_\nu^{(j)}(x)=D^jP_{\nu i}(x),\ \text{for}\ x\in Q_{\nu i}\cap F\ \text{and}\ P_{\nu i}=P_{Q_{\nu i}},$$

where $P_{Q_{\nu i}}$ is the polynomial of degree $\leq[\alpha]$ introduced in Theorem 1.2. We shall prove (1.2)—(1.5) in Proposition 1.1. From the definition of $f_\nu^{(j)}$ and (1.6) we find, for $|j|\leq k$, if $x\in Q_{\nu i}\cap F$,

$$|f^{(j)}(x)-f_\nu^{(j)}(x)|=|f^{(j)}(x)-D^jP_{\nu i}(x)|\leq c2^{-\nu(\alpha-|j|)},$$

i. e. (1.2) holds. Similarly, we obtain (1.3), if we assume that $|j|=\alpha=k+1$ and apply (1.7) with $Q=Q_{\nu i}$ and $Q'=Q_{\nu+1,s}$ for suitable i and s, $x\in F\cap Q_{\nu i}\cap Q_{\nu+1,s}$. In order to prove (1.4) we assume that $|x-y|\leq a2^{-\nu}$. Then, for $x\in Q_{\nu i}\cap F$ and $y\in Q_{\nu s}\cap F$, there exists a cube $Q\supset Q_{\nu i}\cup Q_{\nu s}$ with side of length $\leq c2^{-\nu}$, and we get for $|j|\leq[\alpha]$, $x, y\in F$, by the definition of R_ν and $f_\nu^{(j)}$:

$$|R_{\nu j}(x, y)|=|D^jP_{\nu i}(x)-\sum_{|j+l|\leq[\alpha]} D^{j+l}P_{\nu s}(y)\frac{(x-y)^l}{l!}|\leq|D^jP_{\nu i}(x)-D^jP_Q(x)|$$

$$+|D^jP_Q(x)-\sum_{|j+l|\leq[\alpha]} D^{j+l}P_Q(y)\frac{(x-y)^l}{l!}|+|\sum_{|j+l|\leq[\alpha]}(D^{j+l}P_Q(y)-D^{j+l}P_{\nu s}(y))\frac{(x-y)^l}{l!}|.$$

Consider the last three terms. The second of these is zero, since P_Q has degree $\leq[\alpha]$; the first and the third are, by (1.6) and (1.7), bounded by $c2^{-\nu(\alpha-|j|)}$. Hence, (1.4) is true. Finally, (1.5) follows from (1.8) and Markov's inequality. Consequently, $\{f^{(j)}\}\in\Lambda_\alpha(F)$ and the proof of Theorem 1.2 is complete.

1.4. A simplified condition. The characterization of $\Lambda_\alpha(F)$ in Theorem 1.2 may be simplified for those sets F, for which the family $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$, $k<\alpha\leq k+1$ is uniquely determined by $f=f^{(0)}$. For general closed sets F the same simplified characterization is applicable for those f for which there exist $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$ such that $f=f^{(0)}$. To get a suitable notation we introduce $\Lambda_\alpha^{(0)}(F)$ as the space of those families $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$ such that $f^{(0)}=0$ and the factor space $\Lambda_\alpha(F)/\Lambda_\alpha^{(0)}(F)$ with the usual norm. The elements in $\Lambda_\alpha(F)/\Lambda_\alpha^{(0)}(F)$ may be identified with functions defined on F.

Theorem 1.3. *A function f, defined on F, belongs to $\Lambda_\alpha(F)/\Lambda_\alpha^{(0)}(F)$ if and only if the following condition holds: For every cube Q in R^n, $Q\cap F\neq\emptyset$, with side of length $\delta>0$ there exists a polynomial P_Q of degree $\leq[\alpha]$ such that*

(1.10) $$|f(x)-P_Q(x)|\leq c\delta^\alpha, \text{ for } x\in Q\cap F,$$

if Q', $Q'\cap F\neq\emptyset$, is a cube with side of length $\delta'>0$, then

(1.11) $$|P_Q(x)-P_{Q'}(x)|\leq c(\max(\delta, \delta'))^\alpha, \text{ for } x\in Q\cap Q',$$

and

(1.12) $$|P_Q(x)|\leq c, \text{ for } x\in Q, \text{ if } \delta=1.$$

The norm in $\Lambda_\alpha(F)/\Lambda_\alpha^{(0)}(F)$ is equivalent to the infimum of the constants c in (1.10)—(1.12).

Before proving the theorem, we note that it and Theorem 1.1 have the following corollary.

Corollary 1.1. *A function f, defined on F, may be extended to R^n to a function in $\Lambda_\alpha(R^n)$ if and only if there exist polynomials P_Q satisfying* (1.10)—(1.12).

Proof of Theorem 1.3. One half of the theorem follows at once: If $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$, then Theorem 1.2 with (1.7) changed to (1.9) shows that there exist P_Q, satisfying (1.10)—(1.12). To prove the other half, assume that f is defined on F and that there exist polynomials P_Q, satisfying (1.10)—(1.12). We shall produce a family $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$ such that $f^{(0)}=f$.

As in part B of the proof of Theorem 1.2 we introduce the nets π_ν, $\nu=1, 2, \ldots$, and define $\{f_\nu^{(j)}\}$, $|j|\leq[\alpha]$, by

$$f_\nu^{(j)}(x)=D^jP_{\nu i}(x), \text{ for } x\in Q_{\nu i}\cap F \text{ and } P_{\nu i}=P_{Q_{\nu i}},$$

where $P_{Q_{\nu i}}$ is the polynomial introduced in Theorem 1.3. Then, for $j=0$' we get by (1.10) that $f_\nu^{(0)}\to f$ on F, as $\nu\to\infty$. Furthermore, for $|j|\leq k$, we get by Markov's inequality and (1.11), if $x\in F\cap Q_{\nu i}\cap Q_{\mu s}$ and $\mu\geq\nu$,

$$|f_\mu^{(j)}(x)-f_\nu^{(j)}(x)|=|D^jP_{\mu s}(x)-D^jP_{\nu i}(x)|\leq c2^{\mu|j|}\max_{Q_{\mu s}}|P_{\mu s}-P_{\nu i}|\leq c2^{\mu|j|}2^{-\nu\alpha}.$$

This gives, if $\mu>\nu$, $x\in F$, and $|j|\leq k$,

$$|f_\mu^{(j)}(x)-f_\nu^{(j)}(x)|=|\sum_{N=\nu}^{\mu-1}(f_{N+1}^{(j)}(x)-f_N^{(j)}(x))|\leq\sum_{N=\nu}^{\mu-1}c2^{|j|(N+1)}2^{-N\alpha}\leq c2^{\nu(|j|-\alpha)}.$$

Hence, $f_\nu^{(j)}(x)$, $\nu=1, 2, \ldots$, is a Cauchy sequence for each $x\in F$ and $|j|\leq k$, converging pointwise to a function, which we denote by $f^{(j)}(x)$; clearly $f^{(0)}=f$. By letting $\mu\to\infty$ in the last formula above we obtain the condition (1.2) in Proposition 1.1. The conditions (1.3)—(1.5) in Proposition 1.1 are now proved as in part B of the proof of Theorem 1.2 with the exception that instead of using (1.6) and (1.7) we use Markov's inequality and (1.11). This completes the proof of Theorem 1.3.

1.5. Markov's inequality and further simplification.

Definition 1.3. *F has the Markov property if, for all polynomials P and all cubes Q with center in F and side of length $\delta\leq 1$,* $\max\{|\operatorname{grad} P(x)|:x\in F\cap Q\}\leq c\delta^{-1}\max\{|P(x)|:x\in F\cap Q\}$, *with a c depending only on F and the degree of P.*

If $F=R^n$, this is the classical Markov inequality; we refer to [2, Section 3] and [7, Section 1] for a detailed study and for examples of sets having this property. The importance of this property is that it guarantees that the technical conditions of type (1.7) and (1.11) may be omitted and that $\{f^{(j)}\}\in\Lambda_\alpha(F)$ is uniquely determined by $f=f^{(0)}$, i. e. that $\Lambda_\alpha^{(0)}(F)$ in Theorem 1.3 is 0; in that case we identify $\{f^{(j)}\}$ with f and speak of $f\in\Lambda_\alpha(F)$.

Theorem 1.4. *Let $F\subset R^n$ be a closed set having the Markov property. Then $\{f^{(j)}\}_{|j|\leq k}\in\Lambda_\alpha(F)$ is uniquely determined by $f=f^{(0)}$. Furthermore, $f\in\Lambda_\alpha(F)$ if and only if $|f(x)|\leq c$ on F and, for every cube Q with center in F and side of length $\delta\leq 1$, there exists a polynomial P_Q of degree $\leq[\alpha]$ such that*

$$|f(x)-P_Q(x)|\leq c\delta^\alpha \text{ for } x\in Q\cap F.$$

The norm in $\Lambda_\alpha(F)$ is equivalent to the infimum of the constants c.

Proof. The theorem follows, if we examine the proof of Theorem 1.3 and remember that because of the Markov property it is in the proof enough, if (1.11) and (1.12) hold for $x\in F\cap Q\cap Q'$ and $x\in F\cap Q$, respectively; in this weaker formulation (1.11) and (1.12) follow from our assumption.

2. The Besov Spaces $B_\alpha^{p,q}(F)$.

2.1. In the papers [3, 4, 6], we introduced Besov spaces $B_\alpha^{p,q}(F)$ on closed sets $F\subset R^n$ of a general type, the so called d-sets. The aim of this section is to show how these spaces may be characterized by means of local polynomial approximation. For the definition of d-sets and the spaces $B_\alpha^{p,q}(F)$ and related material we refer to [6] or [2]. We recall here only that d-sets have Hausdorff-dimension d, that the restriction to a d-set F of the d-dimensional Hausdorff-measure is in a natural way associated to F (below we denote this measure by μ), and that the elements of $B_\alpha^{p,q}(F)$ are families $\{f^{(j)}\}_{|j|\leq k}$ of functions defined μ-a. e. on F. Proofs of the two last theorems below may be found in [2].

Our main motivation for the study of the spaces $B_\alpha^{p,q}(F)$ is the following trace theorem, which was proved in [6], where a more precise statement is given.

Theorem 2.1. *Let F be a d-set, $0<d\leq n$, $1\leq p, q\leq\infty$, $\beta=\alpha-(n-d)/p>0$. Then $B_\beta^{p,q}(F)$ is the trace of $B_\alpha^{p,q}(R^n)$ to F.*

The spaces $\Lambda_\alpha(F)$, discussed in the previous section, are related to the Besov spaces as follows (see [2]).

Proposition 2.1. *Let F be a d-set and $\alpha>0$. Then $\Lambda_\alpha(F)=B_\alpha^{\infty,\infty}(F)$ with equivalent norms. Here, an element $\{f^{(j)}\}_{|j|\leq k}\in B_\alpha^{\infty,\infty}(F)$ belongs to $\Lambda_\alpha(F)$ in the following sense: The functions $f^{(j)}$ may in a unique way be altered on a set of μ-measure zero, so that they become continuous and $\{f^{(j)}\}_{|j|\leq k}$ belongs to $\Lambda_\alpha(F)$.*

2.2. Characterization of $B_\alpha^{p,q}(F)$ by means of local polynomial approximation. From now on, π denotes a subdivision of R^n into equally big cubes Q with side length r, half-open in the form $\{x|a_i<x_i\leq a_i+r,\ i=1, 2, \ldots, n\}$, obtained by intersecting R^n with hyperplanes orthogonal to the axes. Such a division is called a *net* π with *mesh* r. $P_k(\pi)$ denotes the set of all functions $m(\pi)$, such that the trace of $m(\pi)$ to a cube Q in π is a polynomial of degree $\leq k$, and $D^j m(\pi)$ is the function, which on Q coincides with the j:th derivative of that polynomial. $\|\ \|_{p,\mu}$ denotes the $L^p(\mu)$-norm.

Theorem 2.2. *Let F be a d-set, $k<\alpha\leq k+1$, $1\leq p, q\leq\infty$. Then $\{f^{(j)}\}_{|j|\leq k}$ belongs to $B_\alpha^{p,q}(F)$ if and only if to every net π with mesh $2^{-\nu}$, $\nu=0, 1, 2, \ldots$, there is a function $m(\pi)\in P_{[\alpha]}(\pi)$ such that for some b_ν with $(\Sigma b_\nu^q)^{1/q}<\infty$ (2.1)–(2.3) hold:*

$$\|f^{(j)}-D^j m(\pi)\|_{p,\mu}\leq 2^{-\nu(\alpha-|j|)}b_\nu,\quad |j|\leq k. \tag{2.1}$$

If $\alpha=k+1$ and π' has mesh $2^{-(\nu+1)}$, then

$$\|D^j m(\pi)-D^j m(\pi')\|_{p,\mu}\leq b_\nu,\quad |j|=k+1. \tag{2.2}$$

If the mesh of π is 1, then

$$\|D^j m(\pi)\|_{p,\mu}\leq b_0,\quad |j|\leq k+1. \tag{2.3}$$

The $B_\alpha^{p,q}(F)$-norm of $\{f^{(j)}\}$ is equal to inf $(\Sigma b_\nu^q)^{1/q}$, *where* inf *is taken over all b_ν, such that* (2.1)—(2.3) *hold for some function $m(\pi)$.*

2.3. If F has the Markov property, the characterization of $B_\alpha^{p,q}(F)$ becomes simpler (c. f. Section 1.5).

Theorem 2.3. *Let F be a d-set with the Markov property, $k<\alpha\leq k+1$, $1\leq p, q\leq\infty$. If $\{f^{(j)}\}_{|j|\leq k}\in B_\alpha^{p,q}(F)$, then the functions $f^{(j)}$ are uniquely determined by $f=f^{(0)}$; we then identify $\{f^{(j)}\}$ with f and speak of $f\in B_\alpha^{p,q}(F)$.*

Furthermore, $f \in B_\alpha^{p,q}(F)$ *if and only if* $f \in L^p(\mu)$ *and to every net* π *with mesh* $2^{-\nu}$, $\nu = 0, 1, 2, \ldots$, *there is a function* $n(\pi) \in P_{[\alpha]}(\pi)$ *such that for some* d_ν *with* $(\Sigma d_\nu^q)^{1/q} < \infty$ $\|f - n(\pi)\|_{p,\mu} \leq 2^{-\nu\alpha} d_\nu$. *The norm in* $B_\alpha^{p,q}(F)$ *is equivalent to* $\|f\|_{p,\mu} + \inf(\Sigma d_\nu^q)^{1/q}$, *where* inf *is taken over all possible choices of* d_ν *and* $n(\pi)$.

The theorem is given in [2] in a less general form, but the proof is the same in this case. In [2] it is also shown that, in general, $B_\alpha^{p,q}(F)$ may not be characterized as in Theorem 2.3.

For $d = n$ a definition of Besov spaces on d-sets by means of local polynomial approximation similar to (and actually equivalent to, see [2]) the characterization of $B_\alpha^{p,q}(F)$ in Theorem 2.3 has been given by Brudnyi (see [1]); he also proves a trace theorem (see also [8]). Now, if $d > n-1$, then F has the Markov property (see [7]), so Theorem 2.3 and Theorem 2.1 give Brudnyi's results.

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