КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'81
София, 1983, с. 376—381

CONSTRUCTIVE FUNCTION THEORY'81
Sofia, 1983, p. 376—381

APPROXIMATION OF PLANE CONVEX COMPACTA BY POLYGONS*

P. S. Kenderov

1. Introduction. In the set CONV of all compact convex subsets of the usual two-dimensional Euclidean plane R^2 we consider the Hausdorff metric $h(A_1, A_2)=\inf\{t>0: A_1\subset A_2+tB, A_2\subset A_1+tB\}$, where $B=\{P\in R^2: |P|\leq 1\}$ is the unit circle and C_1+C_2 stands for the Minkowski sum of two sets $C_1, C_2\in$ CONV. For every integer $n\geqq 3$ we denote by POLY_n the set of all convex polygons with not more than n-vertices. The elements of POLY_n will be called n-gons. The n-gon Δ_0 is said to be a best Housdorff approximation in POLY_n for the set $A\in$ CONV, if $h(A, \Delta_0)=\inf\{h(A, \Delta): \Delta\in \mathrm{POLY}_n\}$. The existence of at least one best Hausdorff approximation for any $A\in$ CONV follows from the well-known Blaschke selection theorem asserting that every bounded sequence of n-gons (n is fixed) contains a subsequence, which is convergent in the Hausdorff metric and tends to some n-gon. In general, as examples like the unit circle or unit square show, the best approximation is not unique. Nevertheless the 'majority' of the elements of CONV have unique best approximation in any POLY_n, $n\geqq 3$. The 'majority' here means 'with an exception of some first Baire category subset of (CONV, h) all convex compact subsets of R^2 have unique best approximation in POLY_n for every $n\geqq 3$' (Theorem 2).

On the way of proving this theorem we give and use a necessary condition for a given $\Delta\in \mathrm{POLY}_n$ to be a best approximation for some $A\in$ CONV. This condition (Theorem 1) coincides with the classical alternating property in the problem of uniform Čebyšev approximation by polynomials. But in our situation this condition is very far from being a sufficient condition for the best approximation.

Some of the results presented here were announced in [5]. The asymptotic behaviour of the sequence $\{r_n(A)\}_{n\geqq 3}$, where $r_n(A)=\inf\{h(A, \Delta): \Delta\in \mathrm{POLY}_n\}$ was studied by Popov [7] and McClure, Vitale [6]. It is an open problem to find necessary and sufficient conditions for a given n-gon Δ to be a best approximation in POLY_n for some $A\in$ CONV. Unknown is also the answer to the following question of Bl. Sendov and

* A part of this work was done while the author was at the University of Frankfurt am Main as a research fellow of the Alexander von Humboldt foundation.

. Popov: Is it true that, among all elements of CONV with fixed perimeter 1, the equilateral $(n+1)$-gon (with the same perimeter) is the worst one to be approximated by n-gons? There is a result of Ivanov [3] concerning approximation by inscribed n-gons which is in favour of the 'yes' answer to the question: among all $(n+1)$-gons with perimeter 1 the equilateral one is the worst to be approximated by inscribed n-gons. In support to the 'yes' answer to this question is also the result from [4].

2. Notations and Results. Let us agree to denote the usual inner product of two points (vectors) $P_1, P_2 \in R^2$ by $\langle P_1, P_2 \rangle$. Put $|P|=\sqrt{\langle P, P \rangle}$. The function (defined in R^2) $s_A(P):=\max\{\langle P, X\rangle: X\in A\}$, where A is a given element of CONV, is called a support function of the set A. This function is positively homogenous and is, therefore, completely determined by its values at the points of the unit circumference $S=\{e\in R^2: |e|=1\}$. $s_A(\cdot)$ is convex and continuous. In this way a mapping $A\to s_A(\cdot)$ is defined from CONV into the space $C(S)$ of all continuous functions in S. It is well-known (and easy to verify) that this mapping is one-to-one (because $A=\{X\in R^2: \langle e, X\rangle \leqq s_A(e),\ e\in S\}$), satisfies $s_{A_1+A_2}=s_{A_1}+s_{A_2}$, $s_{tA}=ts_A$, $t\geqq 0$, and is isometric: for $A', A'' \in$ CONV $h(A', A'')=\max\{|s_{A'}(e)-s_{A''}(e)|: e\in S\}$. Because of this fact the problem of approximating the elements of CONV by elements of POLY$_n$, $n\geqq 3$, with respect to the Hausdorff metric is equivalent to the approximation of the support functions of elements of CONV by support functions of n-gons relative to the uniform norm in $C(S)$. In what follows we identify CONV and POLY$_n$ with their images in $C(S)$ under the above defined mapping.

Further we need an orientation on S. We take the counterclockwise direction on S as positive. For $e_1, e_2 \in S$ we denote by $[e_1, e_2]$ the arc on S with end points e_1 and e_2, which connects e_1 with e_2 in the positive direction. Thus $(e_2, e_1)=S\setminus[e_1, e_2]$. As with segments, by (e_1, e_2) we denote the 'open' arc, i. e. $[e_1, e_2]$ without the end points e_1 and e_2. It is clear what $[e_1, e_2)$ and $(e_1, e_2]$ mean. When there is no danger of ambiguity, the symbol $[e_1, e_2]$ (or $[e_1, e_2)$, (e_1, e_2), $(e_1, e_2]$) will denote the length of the corresponding arc as well.

Let $\Delta=(P_1, P_2, \ldots, P_n)$ be an n-gon with vertices P_i, $i=1, 2, \ldots, n$ and let $e_i\in S$, $i=1, 2, \ldots, n$ be the side-direction of P_iP_{i+1} $(P_{n+1}=P_1)$, i. e. e_i is perpendicular to P_iP_{i+1} and points outward Δ. It is easily seen that for $e\in[e_i, e_{i+1}]$, $S_\Delta(e)=\langle e, P_{i+1}\rangle$. Then $h(A, \Delta)=\max\{|s_A(e)-s_\Delta(e)|: e\in S\}=\max\{\max\{|s_A(e)-\langle e, P_{i+1}\rangle|: e\in[e_i, e_{i+1}]\}\ i=1, 2, \ldots, n\}$.

Therefore, in order to study the best approximation of A by n-gons, we have to investigate the properties of the function $s_A(e)-\langle e, P_{i+1}\rangle$ in $[e_i, e_{i+1}]$. These properties are collected in the following result.

Let $M\in R^2\setminus A$, $A\in$ CONV. Put $d(M, A)=\min\{|X-A|: X\in A\}$. By the strict convexity of the Euclidean norm $|\cdot|$ there exists just one point $N\in A$ such that $|M-N|=d(M, A)$. Put $e^*=(M-N)/|M-N|$. It is clear that $s_A(e^*)=\langle e^*, N\rangle$ and $\langle e^*, M\rangle=s_A(e^*)+d(M, A)$.

In order to simplify the things, we assume that A has interior points, i. e. is non-degenerated.

Proposition 1. There exists a unique vector $e'\in(e^*, -e^*)$ such that

a) $s_A(e')-\langle e', M\rangle=d(M, A)$;

b) when e runs from e^* to e' in the positive direction, the function $s_A(e)-\langle e, M\rangle$ strictly increases from $-d(M, A)$ (for $e=e^*$) to $d(M, A)$ (for $e=e'$);

c) for $e \in (e', -e^*]$ $s_A(e) - \langle e, M\rangle > d(M, A)$.

Analogously, there exists a uniquely determined vector $e'' \in (-e^*, e^*)$ such that $s_A(e) - \langle e, M\rangle$ strictly increases from $-d(M, A)$ (for $e = e^*$) to $d(M, A)$ (for $e = e''$), when e runs in the negative direction on S. For $e \in (-e^*, e'')$ $s_A(e) - \langle e, M\rangle > d(M, A)$.

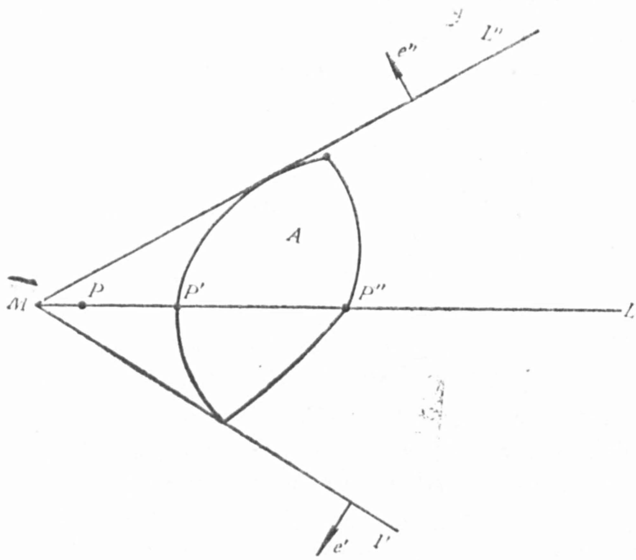

Fig. 1

Corollary 1. *$\max\{|s_A(e) - \langle e, M\rangle| : e \in [e'', e']\} = d(M, A)$ and $|s_A(e) - \langle e, M\rangle| > d(M, A)$ for $e \in (e', e'')$.*

We need further an operation which plays an important role in our considerations. To each pair $e'', e' \in S$, $0 < [e'', e'] < \pi$, we assign a point $M = M(A;\ e'', e')$, a number $d_0 = d_0(A;\ e'', e')$ and a vector $e^* \in (e'', e')$ so that

i) $s_A(e'') - \langle e'', M\rangle = d_0$,

ii) $s_A(e') - \langle e', M\rangle = d_0$,

iii) $s_A(e^*) - \langle e^*, M\rangle = -d_0$,

iv) $\max\{|s_A(e) - \langle e, M\rangle| : e \in [e'', e']\} = d_0$.

This is done as follows. Consider the lines $L'' = \{X \in R^2 : s_A(e'') = \langle e'', X\rangle\}$ and $L' = \{X \in R^2 : s_A(e') = \langle e', X\rangle\}$. Since $0 < (e'', e') < \pi$, there exists only one intersection point $\bar{M}$, i. e. $\langle \bar{M}, e'\rangle = s_A(e')$, $\langle \bar{M}, e''\rangle = s_A(e'')$ (see Fig. 1). There are two possibilities:

a) $\bar{M} \in A$. In this case we put $d_0 = 0$, $M = \bar{M}$ and take e^* arbitrarily in (e'', e'). All the requirements are fulfilled because of the following simple fact.

Lemma 1. *If for $i = 1, 2$ and $A \in \mathrm{CONV}$ $s_A(e_i) = \langle e_i, M\rangle$, where $0 < (e_1, e_2) < \pi$ and $M \in A$, then $s_A(e) = \langle e, M\rangle$ for each $e \in [e_1, e_2]$.*

b) $\bar{M} \notin A$. Then $d(\bar{M}, A) > 0$. Consider the bisector line $L = \{X \in R^2 : \langle e'' - e', X\rangle = \langle e'' - e', \bar{M}\rangle\}$ passing through $\bar{M}$ and intersecting the set A (Fig. 1). The intersection of L with A is the segment $[P', P'']$. When a point P from L moves from $\bar{M}$ toward P', the function $d(P, A)$ decreases from $d(\overline{M}, A)>0$ to $0=d(P', A)$. At the same time the function $f(P)=s_A(e')-\langle e', P\rangle=s_A(e'')-\langle e'', P\rangle$ increases from $0=f(\overline{M})$ to $f(P')>0$. Hence, on the line L there exists just one point M between $\overline{M}$ and P', for which $f(M)=d(M, A)$. This point M, $e^*=(M-N)/|M-N|$, where $N\in A$ and $|M-N|=d(M, A)=\min\{|M-N|: X\in A\}$ and $d_0:=d(M, A)$ satisfy i), ii) and iii). Proposition 1 and Corollary 1 imply that iv) is also fulfilled.

The next result reveals one important extremal property of this construction. It shows that in $[e'', e']$ the function $\langle e, M\rangle$ approximates $s_A(e)$ better than any other function of the type $\langle e, P\rangle$.

Proposition 2. Let $A\in$ CONV, $P\in R^2$ and $e_1, e_2\in S$, $0<(e_1, e_2)<\pi$. Put $d_i=s_A(e_i)-\langle e_i, P\rangle$, $i=1, 2$, and $d_3=\max\{\langle e, P\rangle-s_A(e): e\in[e_1, e_2]\}$. If for some pair of unit vectors e'', e', $0<(e'', e')<\pi$, we have

v) $[e'', e']\subset[e_1, e_2]$;

vi) $d_0\geqq\max\{d_i: i=1, 2, 3\}$,

then $d_0=d_1=d_2=d_3$ and $P=M(A; e'', e')$.

Proof. Let us first consider the case when $d_0=0$. I. e. the point $M=M(A; e'', e')$, satisfying $\langle e'', M\rangle=s_A(e'')$, $\langle e', M\rangle=s_A(e')$, belongs to A. In this case M lies on L' and L''. Since $d_3\geqq-d_i$, $i=1, 2$, we have $0=d_0\geqq\max\{d_1, d_2, d_3\}\geqq\max\{\pm d_i: i=1, 2\}\geqq0$. Hence $d_i=0$, $i=1, 2, 3$. In other words, $s_A(e_i)=\langle e_i, P\rangle$, $i=1, 2$, and $\langle e, P\rangle\leqq s_A(e)$ for $e\in(e_1, e_2)$. The following lemma implies that $P\in A$.

Lemma 2. *Let* $P\notin A\in$ CONV *and let, for some* $e_i\in S$ $i=1, 2$, $0<(e_1, e_2)<\pi$, $s_A(e_i)=\langle e_i, P\rangle$ $i=1, 2$. *Then*

a) $s_A(e)<\langle e, P\rangle$ *for* $e\in(e_1, e_2)$,

b) $s_A(e)>\langle e, P\rangle$ *for* $e\in(e_2, e_1)=S\setminus[e_1, e_2]$.

Once we know that $P\in A$, we get from Lemma 1 that $\langle e, P\rangle=s_A(e)$ for every $e\in[e_1, e_2]$. In particular, taking $e=e'$ or $e=e''$ and remembering v), we see that P also lies on the lines L' and L''. Therefore $M=P$.

Let now consider the case $d_0>0$. Put again $M=M(A; e'', e')$ and $d_0=d_0(A; e'', e')$. By Proposition 1, c), we know that $s_A(e)-\langle e, M\rangle>d_0$ for $e\in S\setminus[e'', e']=(e', e'')$. Since $[e'', e']\subset[e_1, e_2]$ we have

vii) $s_A(e_i)-\langle e_i, M\rangle\geqq d_0\geqq d_i=s_A(e_i)-\langle e_i, P\rangle$, $i=1, 2$.

Then

viii) $\langle e_i, P-M\rangle\geqq0$, $i=1, 2$.

Since $0<(e_1, e_2)<\pi$, we get from here that $\langle e, P-M\rangle\geqq0$ for each $e\in(e_1, e_2)$. In particular, for $e^*\in[e'', e']\subset[e_1, e_2]$ we have $\langle e^*, P\rangle\geqq\langle e^*, M\rangle$. Then

ix) $d_0=\langle e^*, M\rangle-s_A(e^*)\leqq\langle e^*, P\rangle-s_A(e^*)\leqq d_3\leqq d_0$.

Therefore $d_0=d_3$. ix) implies also that everywhere in vii), viii) and ix) we have equalities and not inequalities. This is possible only if $d_0=d_1=d_2$, $M=P$ and $e_1=e''$, $e_2=e'$.

Corollary 2. *If not all of the numbers* d_1, d_2, d_3 *are equal, then* $d_0<\max(d_1, d_2, d_3)$.

Definition 1. Let Δ be an n-gon with vertices $M_1, M_2, \ldots, M_n$ and side-directions $e_1, e_2, \ldots, e_n$, ordered according to the positive direction on S. Δ is said to be alternating for $A\in$ CONV, if

a) $h(A, \Delta)=s_A(e_i)-s_\Delta(e_i)$, $i=1, 2, \ldots, n$,

b) there exists $e_i^*\in(e_{i-1}, e_i)$, $i=1, 2, \ldots, n$ $(e_0=e_n)$ such that $h(A, \Delta)=s_\Delta(e_i^*)-s_A(e_i^*)$, $i=1, 2, \ldots n$.

Theorem 1. *Let $A \in \text{CONV}$ have interior points, $n \geq 3$, and Δ be a best Hausdorff approximation for A in POLY_n. Then Δ is alternating for A.*

Proof. Let $\Delta = (P_1, P_2, \ldots, P_n)$ be a best approximation in POLY_n for A and let $e_1, e_2, \ldots, e_n$ be the side-directions of Δ. Put $d = h(\Delta, A)$.

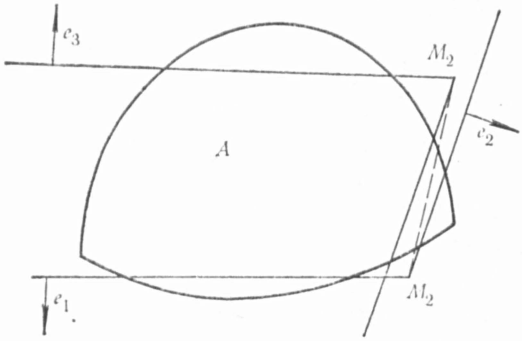

Fig. 2

Suppose that at least one of the numbers (called from now on 'characteristic numbers') $s_A(e_i) - \langle e_i, P_i \rangle$, $\max\{\langle e, P_i \rangle - s_A(e) : e \in [e_{i-1}, e_i]\}$, $i = 1, 2, \ldots, n$ is not equal to d. For instance, suppose one of the three numbers $d_1 = s_A(e_1) - \langle e_1, P_2 \rangle$, $d_2 = s_A(e_2) - \langle e_2, P_2 \rangle$, $d_3 = \max\{\langle e, P_2 \rangle - s_A(e) : e \in [e_1, e_2]\}$ is strictly less than d. We will construct now another n-gon Δ' such that $h(A, \Delta') \leq d$ and for which the corresponding numbers d_1', d_2', d_3' are strictly less than d. We put $M'_{i+1} = M(A; e_i, e_{i+1})$, $i = 1, 2, \ldots, n$ $(e_{n+1} = e_1, M'_1 = M'_{n+1})$.

From Proposition 2 we get $d_0(A; e_1, e_2) < \max(d_1, d_2, d_3) \leq d$ and $d_0(A; e_i, e_{i+1}) \leq d$, $i = 2, 3, \ldots, n$. Using Proposition 1 and Fig. 2, it is not difficult to realize that $h(A, \Delta') \leq d$. Applying the same construction once more (this time to Δ'), we obtain another n-gon Δ'' such that

1) $h(A, \Delta'') \leq h(A, \Delta') \leq d$,

2) at least two more of the characteristic numbers are strictly less than d. After a finite number of similar steps we shall get to an n-gon $\bar{\Delta}$ all characteristic numbers of which are strictly less than d and therefore $h(A, \bar{\Delta}) < d$. This contradicts the choice of Δ as a best Hausdorff approximation for A in POLY_n.

Theorem 2. *The set of all those $A \in \text{CONV}$, which have unique best Hausdorff approximation in POLY_n for every $n \geq 3$ contains a dense G_δ subset of (CONV, h). I. e. the set $\{A \in \text{CONV} : A$ has more than one best approximation in at least one $\text{POLY}_n, n \geq 3\}$ is of the first Baire category in (CONV, h).*

One way to prove this assertion is given in the paper of Gruber, Kenderov [2]. Another way is outlined in [5]. We suggest here an improved argument which is based on the following observation made also by N. Živkov.

Proposition 3. Let Δ be a best approximation for A in POLY_n. Then the set $tA + (1-t)\Delta$, $0 < t < 1$, has unique best approximation in POLY_n.

The proof of this proposition is based on Theorem 1.

The proof of Theorem 2 runs now as follows.

Consider the metric projection π_k: CONV $\rightarrow$ POLY$_k$ assigning to each $A \in$ CONV the set $\pi_k(A)$ of all best approximations for A in POLY$_k$. As it follows from a result of Singer [8], π_k: (CONV, h) $\rightarrow$ (POLY$_k$, h) is an upper semi-continuous mapping. By a theorem of Fort [1] there exists a dense G_δ subset W_k of (CONV, h), at the elements of which π_k is lower semicontinuous, i. e. for every $A \in W_k$, $\varepsilon > 0$ and $\Delta \in \pi_k(A)$ there exists $\delta > 0$ such that any $A' \in$ CONV with $h(A', A) < \delta$ has at least one best approximation Δ' such that $h(\Delta, \Delta') < \varepsilon$. We shall show now that such an element A of W_k has unique best approximation. Indeed, suppose $\Delta_1, \Delta_2 \in \pi_k(A)$, $\Delta_1 \neq \Delta_2$ and take $\varepsilon = h(\Delta_1, \Delta_2)/2$. Consider the sets $A_1(t) = tA + (1-t)\Delta_1$ and $A_2(t) = tA + (1-t)\Delta_2$. According to Proposition 3 $A_i(t)$ has only Δ_i as best approximation in POLY$_k$, $i = 1, 2$. As $\lim_{t \to 1} h(A, A_i(t)) = 0$, this contradicts the lower semicontinuity of π_k at A. The set $W = \bigcap_{k=1}^{\infty} W_k$ is the desired dense and G_δ subset of (CONV, h). Theorem 2 is proved.

REFERENCES

1. M. K. Fort. Points of continuity of semi-continuous functions. *Publ. Math. Debrecen*, **2**, 1951, 100-102.
2. P. Gruber, P. Kenderov. Approximation of convex bodies by polytops. *Rend. Ciricolo Mat. Palermo, Ser.* 2, **31**, 1982, 195-225.
3. Р. Иванов. Апроксимация на изпъкнали n-ъгълници с вписани $(n-1)$-ъгълници. — В: Математика и математическо образование. София, 1973, 115-122.
4. П. С. Кендеров. Об одном оптимальном свойстве квадрата. *Доклады БАН*, **26**, 1973, 1143-1146.
5. P. S. Kenderov. Approximation of plane convex compacta by polygons. *C. R. Acad. bulg. Sci.*, **33**, 1980, 889-891.
6. D. E. McClure, R. A. Vitale. Polygonal approximation of plane convex bodies. *J. Math. Anal. Appl.*, **51**, 1975, 326-358.
7. В. Попов. Апроксимация на изпъкнали множества. *Изв. мат. инст. БАН*, **11**, 1970, 67-79.
8. I. Singer. Some remarks on approximative compactness. *Rev. Roum. Math. Pures et Appl.*, **9**, 1964, 167-177.

Centre of Mathematics and Mechanics
1090 Sofia P. O. Box 373 Bulgaria

Received on June 21, 1981