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# CONVERGENCE OF INTERPOLATING RATIONAL FUNCTIONS

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***Summary.*** Let $E$ be a regular compact set in $\boldsymbol{C}$, $\{\beta_k\}_{k=1}^{\infty}=\beta\in\mathscr{P}(E)$ and $f$ be an analytic (and single-valued) function on $E$. For each integer $n$ let $\mathscr{R}_n^{\beta}(f)=\mathscr{R}_n$ denote the interpolating rational function to $f$ of order $n$ with respect to $\beta$. Assume the function $f$ and all the functions $\mathscr{R}_n$, for $n$ sufficiently large, are analytic in some region $E_R$, $R>1$ ($f\in\mathscr{H}_*^*(E_R)$). There is considered the question how large must be $R$ such that the sequence $\mathscr{R}_n$ converges to $f$, as $n=1, 2, \ldots$ uniformly on the set $E$ (Theorem 1). If $R$ is large enough ($R>R_2$), we set $\rho_f(E_R)=\sup\{\rho, \mathscr{R}_n\to f$ as $n\to\infty$, uniformly on each compact subset of the region $E_\rho\}$ and $\inf\{\rho_f(E_R), f\in\mathscr{H}_*^*(E_R)\}=\rho_R(E_R)$. The second problem considered in the paper is to give a lower and an upper estimate of $\rho(E_R)$.

Let $E$ be a compact set in $\boldsymbol{C}$, whose complement $K$ (with respect to the extended plane $\bar{\boldsymbol{C}}$) is connected and regular in the sense that $K$ possesses Green's function $G(z, \infty)$ with a pole at infinity. We set $G(z, \infty)\equiv 1$ for $z\in E$. For each $r>1$ let $\Gamma_r$ denote generically the locus $\{z, G(z, \infty)=\log r\}$ and $E_r$ — the region $E\cup\{z, G(z, \infty)<\log r\}$. We shall assume without losing the generality that $0\in E$. It is well-known that there exists a sequence $\beta$ of points $\beta_1, \beta_2, \ldots$, lying on $E$ and satisfying the relation

$$\lim_{n\to\infty}|\omega_n^{\beta}(z)|^{1/n}=\lim\Big|\prod_{k=1}^{n}(z-\beta_k)\Big|^{1/n}=C\,.\exp G(z, \infty)$$

uniformly on each compact subset of $K$; $C$ is a positive constant. The sequence $\beta$ is said to be an element of $\mathscr{P}(E)$ (concerning the existence of sequences $\beta$ with the indicated properties see [1, Section 7], [2, Section 5]).

Assume the function $f$ is analytic (and single-valued) on $E$. (This means that $f$ is analytic in some open region of $E$.) We'll write $f\in\mathscr{H}(E)$. Let $(n, m)$ be a fixed pair of integers. Define the rational function $\mathscr{R}_{n,m}^{\beta}(f)=\mathscr{R}_{n,m}=p/q$, $q\neq 0$ in the following way. The numerator $p$ and the denominator $q$ are polynomials of degree $\leq n$ and $\leq m$, respectively ($\deg p\leq n$, $\deg q\leq m$), and are determined so that the function $\varphi=(f\cdot q-p)/\omega_{n+m+1}^{\beta}$ is analytic on $E$. It is easy to verify that the rational function $\mathscr{R}_{n,m}$ always exists and is the unique one, for which the numerator and the denominator satisfy the upper requirement. This construction is due to S a f f [3].

For each pair $(n, m)$ we set $\mathscr{R}_{n,m}=\mathscr{P}_{n,m}/\mathscr{Q}_{n,m}$, where $\mathscr{Q}_{n,m}(z)=1+\cdots$ and the polynomials $\mathscr{P}_{n,m}$ and $\mathscr{Q}_{n,m}$ have not a common divisor. The zeros of $\mathscr{Q}_{n,m}$ are called free poles of the rational function $\mathscr{R}_{n,m}$. If $\mathscr{R}_{n,m}$ has exactly $m$ free poles, then it interpolates $f$ in all the points $\beta_1, \ldots, \beta_{n+m+1}$.

Suppose the function $f$ can be analytically extended in some region $E_R$, $R>1$. Let $\Lambda=(n, m_n)_{n=1}^{\infty}$ be a sequence of pairs with the property $n/m_n \to 1$ as $n=1,2,\ldots$; set for $n=1, 2 \ldots \mathscr{Q}_{n,m_n}=\mathscr{Q}_n$, $\mathscr{R}_{n,m_n}=\mathscr{R}_n$ and suppose that, for all $n$ sufficiently large $(n\geq N)$, $\deg \mathscr{Q}_n=m_n$ and $\mathscr{Q}_n(z)\neq 0$, if $z\in E_R$ (we'll write $\mathscr{R}_n \in \mathscr{H}_*(E_R)$, $f\in \mathscr{H}^*_{*\Lambda}(E_R)$). The problem is to determine $R$ such that the sequence $\mathscr{R}_n$ converges to $f$, as $n\to\infty$, uniformly on the set $E$. It can be proved the following

Theorem 1. *Let $E$ be a regular set in $\mathbf{C}$ with a regular complement $K$, $\beta\in\mathscr{P}(E)$ and $f$ be analytic function on $E$. Let $\Lambda=(n, m_n)_{n=1}^{\infty}$ be a sequence of pairs of integers with the property $n/m_n\to 1$ as $n=1,2,\ldots$. Suppose that for some $R$, $R>1$, $f\in\mathscr{H}^*_{*\Lambda}(E_R)$. Then, if $R>R_1$, where $R_1$ is the largest positive zero of the equation*

$$\mathscr{A}(x)=x^6-4x^5-12x^4-4x^3-11x^2-2=0,$$

*the sequence $\mathscr{R}_n$ converges to $f$, as $n\to\infty$, uniformly on $E$* (concerning the proof of Theorem 1 see [4]).

Theorem 1 has the following

Corollary. *In the conditions of Theorem 1 the sequence $\mathscr{R}_n$ converges to $f$, as $n\to\infty$, uniformly on each compact subset of the region $E_{(R/R_1)}$* (see [4]).

Let now suppose without losing the generality that $\operatorname{Cap} E=1$. Denote by $R_2$ the bigger positive zero of the equation $\mathscr{R}(x)=x^2-7x+1=0$. We set for each $a\in E_R$

$\rho(a)=\min\{\exp G(z,\infty)\Rightarrow |z-a|=1\}$. It can be verified that $\rho(a)>1$, if $R>R_2$ $(a\in\Gamma_R)$. In this case $(R>R_2)$, the point $a$ lies exterior to $\Gamma_\rho(a)$ and $\operatorname{dist}(a, \Gamma_{\rho(a)})=1$. We have the following

Theorem 2. *Let $E$ and $\beta$ be given as in Theorem 1, $\operatorname{Cap} E=1$ and $R$ be a positive number, $R>R_2$. Then there exists for each $a\in\Gamma_R$ a function $f_a$, $f_a\in\mathscr{H}(E_R)$ and a sequence of pairs of integers $\overline{\Lambda}=(n_k, m_{n_k})_{k=1}^{\infty}$ with the property $n_k\to\infty$, $m_{n_k}\to\infty$ as $k=1,2,\ldots$ and $n_k/m_{n_k}\to 1$, $k=1, 2,\ldots$, such that all the functions $\mathscr{R}_{n_k}(f_a)=\mathscr{R}^{\beta}_{n_k,m_{n_k}}(f_a)$ belong to $\mathscr{H}_*(E_R)$ and $\mathscr{R}_{n_k}\to\infty$, $k=1,2,\ldots$, uniformly in the region $e(a)=E_R\cap\{z, |z-a|<\rho(a)/R^2\}$ $(\mathscr{R}_{n_k}\rightrightarrows\infty$ in $e(a))$.*

The used construction of the function $f_a$, $a\in\Gamma_R$ in Theorem 2 and of the rational functions $\mathscr{R}_{n_k}$, $k=1,2,\ldots$, is due to Rahmanov [5].

There are two problems in connection with Theorem 1 and Theorem 2. We assume first that $\Lambda=(n, m_n)_{n=1}^{\infty}$ is a sequence with the indicated property $(n/m_n\to 1$, as $n\to\infty)$; we set $\mathscr{R}^{\beta}_{n,m_n}=\mathscr{R}_n$ $(\beta\in\mathscr{P}(E))$. Let us now define the class $W(E)$ in the following way:

$$W_\Lambda(E)=\{f\in\mathscr{H}(E)\Rightarrow\mathscr{R}_n\rightrightarrows f,\quad n\to\infty \text{ on } E\}.$$

We set

$$R_0(E)=\inf\{R,\ R>1\Rightarrow \mathscr{H}^{*}_{*\Lambda}(E_R)\subset W(E)\}.$$

There follows from Theorem 1, that $R_0(E)\leqq R_1$. The first problem is to give a more exact upper estimate of $R_0(E)$ ($E$ is a compact given as in Theorem 1).

The problem of giving a lower estimate of $R_0(E)$ can be connected with Theorem 2. Consider first the following scheme.

Suppose $\mathscr{H}^{*}_{*\Lambda}(E_R)\subset W_\Lambda(E)$ for some $R$, $R>1$ and let $f\in\mathscr{H}^{*}_{*\Lambda}(E_R)$. We set $\rho_f(E_R)=\sup\{\rho,\ R_n\rightrightarrows f$ as $n\to\infty$, on each compact subset of $E\}$.

Denote by $\rho_R(E)=\inf\{\rho_f(E_R),\ f\in\mathscr{H}^{*}_{*}(E_R)\}$. There follows from Theorem 1 that $\rho_R(E)\geqq R/R_1$, if $R>R_1$. We have from Theorem 2, in case $R>R_2$ and $\operatorname{Cap} E=1$, the estimate $\rho_R(E)\leqq\rho(E)$, where $\operatorname{dist}(\Gamma_R,\Gamma_{\rho(E)})=1/R_2^2$. The requirement $\operatorname{Cap} E=1$ in Theorem 2 doesn't lose the generality, it helps only to simplify some calculations and to write the results more simply. It follows from Theorem 2 that there exist compacts $E$ and $\beta\in\mathscr{P}(E)$ such that if $\operatorname{dist}(E,\Gamma_R)\leqq R_2^{-2}$, then there exists a function $f\in\mathscr{H}(E_R)$ so that $R_n$ doesn't converge to $f$, $n=1, 2,\dots$, on the set $E$.

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