КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'81
София, 1983, с. 385—388

CONSTRUCTIVE FUNCTION THEORY'81
Sofia, 1983. p. 385—388

DIRECT AND INDIRECT JACKSON'S TYPE THEOREMS IN SOME FRECHET FUNCTIONS SPACES*

G. Kozłowska

Summary. In this paper some direct and indirect Jackson's type theorems are proved, in which functions from a Fréchet space are approximated by trigonometric polynomials.

1. Introduction. Direct and indirect Jackson's type theorems in the spaces $L^p(0<p<1)$ were investigated by Ivanov [2]. Generalized Ivanov's results in Fréchet space are proved in this paper.

Let $\Phi(s)$ be a continuous and nondecreasing function, defined for $s\geqq 0$, such that $\Phi(0)=0$, $\Phi(s)>0$ for $s>0$ and $\Phi(s_1+s_2)\leqq\Phi(s_1)+\Phi(s_2)$.

Φ space is called a linear space L^Φ, measurable over A (A is a set in R^1) and finite almost everywhere functions f such that $//f//=\int_A \Phi(|f(x)|)\,dx<\infty$ with metric $\rho(f,\varphi)=//f-\varphi//$ $(f,\varphi\in L^\Phi)$.

Every Φ space is a Fréchet space with the F-norm $//f//$ ([1, p. 584]).

Let $G=\{g_k\}$ be linear independent system functions from L^Φ and $f\in L^\Phi$.

Let's denote the best approximation of $f\in L^\Phi$ by polynomials of degree $\leqq n$ from G system by

$$E_n(f,G)_\Phi=E_n(f)_\Phi=\inf_{\{a_k\}}//f-\sum_{k=0}^{n}a_kg_k//.$$

$E_n(f)_\Phi$ of course decreases with respect to n.

Lemma 1. *For any function $f\in L^\Phi$ there exists an element of the best approximation $P_n=\Sigma_{k=0}^n c_kg_k$, i. e. $E_n(f)_\Phi=//f-P_n//$.*

The lemma is a consequence of the theorem, p. 590 [1].

It is easily seen that the Lemma 2 is true.

Lemma 2. *If $f_i\in L^\Phi$ for $i=1, 2$, then $E_n(f_1+f_2)_\Phi\leqq E_n(f_1)_\Phi+E_n(f_2)_\Phi$.*

When $\delta>0$ and $f\in L^\Phi$, then $\omega_k(\delta,f)_\Phi=\sup_{|h|\leqq\delta}//\Delta^k{}_hf(x)//$, where $\Delta_h^kf(x)=\Sigma_{i=0}^k(-1)^{k+i}C_k^if(x+ih)$.

It is easily seen that Lemmas 3 and 4 are true, too.

---

* First modified form of this paper will be published in *Zeszyty Naukowe Politechniki Śląskiej, Matematyka — Fizyka*, **39**.

Lemma 3. *Let $f\in L^{\Phi}$, then for $\delta\geqq 0$ and $\eta\geqq 0$ $0=\omega(0, f)_{\Phi}\leqq\omega(\delta, f)_{\Phi}\leqq\omega(\delta+\eta, f)_{\Phi}\leqq\omega(\delta, f)_{\Phi}+\omega(\eta, f)_{\Phi}$.*

Hence, if $n\in N$, we get $\omega(n\delta, f)_{\Phi}\leqq n\omega(\delta, f)_{\Phi}$.

Lemma 4. *Let $f, g\in L^{\Phi}$, then $\omega_k(\delta, f+g)_{\Phi}\leqq\omega_k(\delta, f)_{\Phi}+\omega_k(\delta, g)_{\Phi}$.*

Lemma 5. *Let f be A-periodic function or $A=(-\infty, +\infty)$, then $\omega_k(\delta, f)_{\Phi}\leqq 2^k\,//f//$.*

Proof. $\omega_k(\delta, f)_{\Phi}=\sup\limits_{|h|\leqq\delta}\int\limits_A\Phi(|\sum\limits_{i=0}^{k}(-1)^{k+i}C_k^i f(x+ih)|)dx$

$$=\int\limits_A\Phi(|\sum\limits_{i=0}^{k}(-1)^{k+i}C_k^i f(x)|)dx\leqq\int\limits_A\Phi(|f(x)|2^k)\,dx\leqq 2^k\int\limits_A\Phi(|f(x)|)\,dx=2^k//f//.$$

2. Approximation in the L^{Φ} Spaces. Let $f\in L^{\Phi}$ be 2π-periodic function and $T_n(x)$ be a trigonometric polynomial of degree $\leqq n\,(n\geqq 0)$.

If $\lim_{t\to\infty}\Phi(t)=\infty$ and $\psi(\alpha)=\sup\{\Phi(\alpha s)/\Phi(s):s\geqq 0\}$ is denoted for $\alpha\geqq 0$, then the Theorems 1 and 2 are true in L^{Φ} space.

Theorem 1. *If $f\in L^{\Phi}\,[0, 2\pi]$ and for any $p\in N\int_1^{\infty}\psi(x^{-p})dx<\infty$, then for $n\geqq 1$ $E_{n-1}(f)_{\Phi}\leqq C_{\Phi}\omega(\pi/n, f)_{\Phi}$ (C_{Φ} is a constant depending on Φ).*

Proof. It is known from the Theorem 3 [2], that there exists a function

$$l_n(x)=c_i,\ x\in[\pi(i-1)/n,\ \pi i/n),\ i=1, \dots, 2n, \text{ such that}$$

(1) $$\int\limits_0^{2\pi}\Phi(|f(x)-l_n(x)|)dx\leqq 4\omega\left(\frac{\pi}{n}, f\right)_{\Phi}.$$

When we denote $c_{2n+1}=c_1$, then

$$\frac{\pi}{n}\sum_{i=1}^{2n}\Phi(|c_{i+1}-c_i|)=\int\limits_0^{2\pi}\Phi(|l_n(x+(\pi/n))-l_n(x)|)\,dx$$

$$\leqq\int\limits_0^{2\pi}\Phi(|\Delta^1_{\pi/n}f(x)|)\,dx+2\int\limits_0^{2\pi}\Phi(|f(x)-l_n(x)|)\,dx.$$

Hence applying (1) we obtain

(2) $$\frac{\pi}{n}\sum_{i=1}^{2n}\Phi(|c_{i+1}-c_i|)\leqq 9\omega(\pi/n, f)_{\Phi}.$$

Let $H_y(x)$ be 2π-periodic Heaviside function for $y\in(0, 2\pi)$, i. e.

$$H_y(x)=\begin{cases}0 & \text{for } 0\leqq x<y,\\ 1 & \text{for } y\leqq x<2\pi.\end{cases}$$

Then ([2, p. 647, Lemma 3]) for every $y\in(0, 2\pi)$, determined $m\in N$ and $n\geqq 1$, there exists a trigonometric polynomial $T_y(x)$ of degree $\leqq(n-1)\,n$ such that for $x\in[0, 2\pi]$

$$|H_y(x)-T_y(x)|\leqq c_m\left\{\left(n\left|\sin\frac{x}{2}\right|+1\right)^{-2m+1}+\left(n\left|\sin\frac{x-y}{2}\right|+1\right)^{-2m+1}\right\}.$$

Of course $l_n(x)=c_1+\sum_{i=1}^{2n-1}H_{x_i}(x)(c_{i+1}-c_i)$ for almost all $x\in[0, 2\pi]$, $x_i=\pi i/n$ $(i=1, \dots, 2n)$.

Let for every n $T_n(x)=c_1+\Sigma_{i=1}^{2n-1} T_{x_i}(x)(c_{i+1}-c_i)$, then

$$\int_0^{2\pi} \Phi(|l_n(x)-T_n(x)|)\,dx \leq \sum_{i=1}^{2n-1} \Phi(|c_{i+1}-c_i|)\int_0^{2\pi} \psi(|H_{x_i}(x)-T_{x_i}(x)|)dx.$$

If $\psi(\alpha)=\sup\{\Phi(\alpha s)/\Phi(s): s>0\}$ then $\psi(\alpha\beta)\leq\psi(\alpha)\psi(\beta)$ and

$$(3)\quad \int_0^{2\pi} \Phi(|l_n(x)-T_n(x)|)dx \leq 2\psi(|c_m|)\sum_{i=1}^{2n} \Phi(|c_{i+1}-c_i|)\int_0^{2\pi} \psi((n\sin\frac{x}{2}+1)^{-2m+1})dx.$$

Because $\psi(\alpha)$ is a nondecreasing function, we get $\int_0^{2\pi}\psi((n\sin\frac{x}{2}+1)^{-2m+1})$ $\times dx\leq 2\pi S_n/n$. We denote by S_n the n-th partial sum of the series $\Sigma_{i=1}^{\infty}\psi(i^{-2m+1})$, where $\psi(i^{-2m+1})\geq 0$ for $i=1,2,\ldots$ and $\psi(1)\geq\psi(2^{-2m+1})\geq\ldots$ Choosing m such that $2m-1\geq p$, $\int_1^{\infty}\psi(x^{-2m+1})dx<\infty$ is obtained and consequently this series is convergent.

Hence $\Sigma_{i=1}^{\infty}\psi(i^{-2m+1})=A_\psi\geq S_n$ and $\int_0^{2\pi}\psi((n\sin\frac{x}{2}+1)^{-2m+1})\,dx\leq 2\pi A_\psi/n$.

Hence by (3) and (2)

$$\int_0^{2\pi} \Phi(|l_n(x)-T_n(x)|)\,dx\leq B_\psi(\pi/n)\sum_{i=1}^{2n}\Phi(|c_{i+1}-c_i|)\leq B_\psi 9\omega(\pi/n, f)_\Phi.$$

Hence by (1)

$$\int_0^{2\pi}\Phi(|f(x)-T_n(x)|)\,dx\leq C_\Phi\omega(\pi/n, f)_\Phi.$$

In that way we get the estimation as desired.

Lemma 6. *Let ψ be such that for any $p\in N$ $\Sigma_{i=1}^{\infty}\psi(i^{-p})<\infty$, then if $k\in N$, $h\in R$ and $n\geq 1$, we get $//\Delta_h^k T_n//\leq C_{\Phi,k}n^k\psi^k(|h|)//T_n//$, where $T_n=T_n(x)$ is a trigonometric polynomial of degree $\leq n$.*

Proof. Let $S_{nl}(x)=(\sin((n+1)x/2)/(n+1)\sin(x/2))^{2l}$, where $l\in N$.

From [2, p. 651] we get

$$|\Delta_h^1 T_n(x)|\leq n(l+1)|h|\{|T_n(x+h)|+\sum_{i=0}^{2n(l+1)} S_{nl}(x_i)|T_n(x+x_i)|\},$$

where $x_i=2\pi i/(2n(l+1)+1)$; $i=0, 1, \ldots, 2n(l+1)$.

Hence $//\Delta_h^1 T_n(x)//\leq n(l+1)\psi(|h|)\{1+\Sigma_{i=1}^{2n(l+1)}\psi(S_{nl}(x_i))\}//T_n//$. Since for $x_i/2\in[0,\pi/2)$ $(n+1)\sin x_i/2\geq(l+1)^{-1}$, then $(\sin((n+1)x_i/2)/(n+1)\sin(x_i/2))^{2l}\leq((l+1)/i)^{2l}$.

Therefore, $\Sigma_{i=1}^{2n(l+1)}\psi[S_{nl}(x_i)]<\psi(1)+2(l+1)^{2l}\Sigma_{i=0}^{\infty}\psi(i^{-2l})$. Choosing $l=l_\psi$ such that the last series is convergent, $\Sigma_{i=0}^{2n(l+1)}\psi[S_{nl}(x_i)]\leq C_\psi$ is obtained. Hence $//\Delta_h^1 T_n(x)//\leq C_\psi^* n\psi(|h|)//T_n//$ and $//\Delta_h^k T_n//\leq n^k C_{\Phi,k}\psi^k(|h|)//T_n//$.

Corollary 1. *Let ψ be such that for any $p\in N$ $\Sigma_{i=0}^{\infty}\psi(i^{-p})<\infty$, then if $k\in N$, $n\geq 1$, we get $\omega_k(\delta, T_n)_\Phi\leq C_{\Phi,k}n^k\psi^k(\delta)//T_n//$.*

Theorem 2. *Let $f\in L^\Phi k$, $n\in N$ and ψ be such that for any $p\in N$ $\Sigma_{i=0}^{\infty}\psi(i^{-p})<\infty$, then*

$$\omega_k(1/n, f)_\Phi\leq C_{\Phi,k}^*\psi^k(\frac{1}{n})\sum_{\nu=0}^{n}(\nu+1)^{k-1}E_\nu(f)_\Phi+2^k E_n(f).$$

P r o o f. Let $t_n(x)$ be a polynomial of the best approximation of degree $\leqq n$, then for integer $m \geqq 0$, from the Lemmas 4 and 5 we obtain

$$\omega_k(1/n, f)_\Phi \leqq 2^k E_{2m+1}(f) + \omega_k(1/n, t_{2m+1})_\Phi. \tag{4}$$

From Corollary 1 we get

$$\omega_k\left(\frac{1}{n}, t_{2m+1}\right)_\Phi \leqq \omega_k\left(\frac{1}{n}, t_{20}-t_0\right)_\Phi + \sum_{\nu=0}^{m} \omega_k\left(\frac{1}{n}, t_{2\nu+1}-t_{2\nu}\right)_\Phi \tag{5}$$

$$\leqq 2C_{\Phi,k}\psi^k\left(\frac{1}{n}\right)\left\{E_0(f) + \sum_{\nu=0}^{m} 2^{(\nu+1)k} E_{2\nu}(f)_\Phi\right\}.$$

Since for $\nu \geqq 1$

$$2^{2k} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{k-1} E_\mu(f)_\Phi \geqq 2^{(\nu+1)k} E_{2\nu}(f)_\Phi.$$

From the above and (5) it comes out that

$$\omega_k\left(\frac{1}{n}, t_{2m+1}\right)_\Phi \leqq C^*_{\Phi,k}\,\psi^k(1/n)\left\{E_0(f)_\Phi + E_1(f)_\Phi + \sum_{\nu=1}^{m} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{k-1} E_\mu(f)_\Phi\right\}$$

$$\leqq C^*_{\Phi,k}\,\psi^k(1/n) \sum_{\nu=0}^{2^m} (\nu+1)^{k-1} E_\nu(f)_\Phi.$$

Choosing m such that $2^m < n \leqq 2^{m+1}$, from the above and (4)

$$\omega_k(1/n, f)_\Phi \leqq 2^k E_{2m+1}(f)_\Phi + C^*_{\Phi,k}\,\psi^k(1/n) \sum_{\nu=0}^{2^m} (\nu+1)^{k-1} E_\nu(f)_\Phi$$

$$\leqq C^*_{\Phi,k}\,\psi^k(1/n) \sum_{\nu=0}^{n} (\nu+1)^{k-1} E_\nu(f)_\Phi + 2^k E_n(f)_\Phi$$

results.

I wish to express my gratitude to Professor Julian Musielak for his valuable suggestions to this paper.

REFERENCES

1. А. Л. Гаркави. Теорема существования элемента наилучшего приближения в пространствах типа (F) с интегральной метрикой. *Мат. заметки*, **8**, 1970, №5, 583-594.

2. В. И. Иванов. Прямые и обратные теоремы теории приближения в метрике L_p для $0 < p < 1$. *Мат. заметки*, **18**, 1975, № 5, 641-658.

Silesian Technical University
Institute of Mathematics
Gliwice Poland

Received on June 2, 1981