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CONVERGENCE THEOREMS FOR INTEGRALS WITH RESPECT TO NON-LINEAR OPERATOR MEASURES

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Summary. In this paper we consider set functions with value in a class of non-linear operators (class $N(S, F)$. We define an integration of totally measurable functions with respect to such non-linear operator measures. Then we introduce a definition of a space $M(\mu)$, which contains also unbounded functions and we extend the domain of definition of the integral to $M(\mu)$. Some examples of such measures and integrals are then given. The rest of the paper is devoted to formulating theorems on absolute continuity of the integral and convergence of the Lebesgue-Vitali type.

Let us recall some definitions, taken from [6].

Definition 1. Let (S, ρ) be a locally compact, metric vector space and F be a Banach space. By $N(S, F)$ we denote the space of all function $U: S \to F$, satisfying the following conditions:

1) $U\,0=0$;

2) U is uniformly continuous on every ball $K_\alpha=\{x \in S,\ \rho(x, 0) \leqq \alpha\}$.

Let X be a nonempty set and $\mathscr{R}$ be a σ-ring of subset of X, $X \in \mathscr{R}$.

Definition 2. A set function $\mu: \mathscr{R} \to N(S, F)$ is said to be an operator measure, if μ has the following properties:

1) $\mu(\emptyset)=0$;

2) for every $r \in S$ and every sequence of pairly disjoint sets $\{B_i\}$ from $\mathscr{R}$ $\mu(\bigcup_{i=1}^{\infty} B_i)r=\sum_{i=1}^{\infty} \mu(B_i)\,r$;

3) for every positive α, δ and every $B \in \mathscr{R}$ the δ-semivariation

$$\mathrm{sv}_\delta(\mu_\alpha, B)=\sup \left\| \sum_{i=1}^{n} \mu(B_i)x_i-\mu(B_i)x_i' \right\| \to 0,\ as\ \delta \to 0,$$

where the supremum is taken over all finite disjoint partitions of B and over all x_i, $x_i' \in K_\alpha$ such that $\rho(x_i, x_i') \leqq \delta$, $1 \leqq i \leqq n$;

4) if for a sequence $\{A_n\} \subset \mathscr{R}$ there exists a sequence $\{B_n\} \subset \mathscr{R}$ such that $B_n \searrow \emptyset$ and $A_n \subset B_n$, then $\mu(A_n)r \to 0$, for every $r \in S$.

By E we shall denote the space of all $\mathscr{R}$-measurable simple functions on X with value in S. Let E_0 denote the space of all uniform limits of functions from E.

Definition 3. Let $\mu:\mathcal{R}\to N(S, F)$ be an operator measure.

a) For every $s\in E$ we define $\int\limits_X s\,d\mu=\Sigma_{i=1}^n \mu(B_i)x_i$, where $s=\Sigma_{i=1}^n x_i\cdot 1_{B_i}$.

b) For every $f\in E_0$ we define

$$\int\limits_X f\,d\mu=\lim_n\int\limits_X s\,d\mu, \text{ where } \{s_n\}\subset E \text{ and } s_n\rightrightarrows f.$$

Such an integration is well defined, this fact can be proved by 3) from Definition 2.

Dinculeanu [3] considered the case of linear operator valued measures. Batt [2] and Friedman, Tong [5] considered non-linear and finitely additive case. We assumed μ to be countably additive in the pointwise sense. Now we shaill give an extension of definition of such integration to the case of the unbounded functions, employing some ideas from [6].

Definition 4. A sequence $\{A_n\}$ of $\mathcal{R}$-measurable sets is said to be μ-convergent to zero ($A_n\xrightarrow{(\mu)}0$) iff for every sequence $\{B_n\}\subset\mathcal{R}$ such that $B_n\subset A_n$ and for every $r\in S$ $\mu(B_n)r\to 0$, as $n\to\infty$.

Definition 5. By $M(\mu)$ we denote the class of all functions f in $M(X, S)$ such that if $A_n\xrightarrow{(\mu)}0$ and ε is given, then for n sufficiently large we have $\|\int f\cdot 1_A\,d\mu\|<\varepsilon$ whenever $A\subset A_n$ and $f\cdot 1_A\in E_0$.

Proposition 6. *If $Z_n\nearrow X$, $f\in M(\mu)$ and $f\cdot 1_{Z_n}\in E_0$, then a sequence $\{\int f\cdot 1_{Z_n}d\mu\}$ is convergent in F.*

For the proof see [7]. Observe that for every $f\in M(X, S)$ there exists a sequence $\{Z_n\}\subset\mathcal{R}$ such that $Z_n\nearrow X$ and $f\cdot 1_{Z_n}\in E_0$. One can take for example $Z_n=f^{-1}(B(0, n))$. As an immediate consequence of the previous fact we get

Proposition 7. *For every $f\in M(\mu)$ there exists a sequence $\{Z_n\}\subset\mathcal{R}$ such that $Z_n\nearrow X$, $f\cdot 1_{Z_n}\in E_0$ and $\{\int f\cdot 1_{Z_n}d\mu\}$ is convergent in F.*

Proposition 8 ([7]). *If $f\in M(\mu)$, $X\setminus W_n\xrightarrow{(\mu)}0$, $X\setminus Z_n\xrightarrow{(\mu)}0$, $f\cdot 1_{W_n}\in E_0$ and $f\cdot 1_{Z_n}\in E_0$, then $\lim_n\int f\cdot 1_{W_n}d\mu=\lim_n\int f\cdot 1_{Z_n}d\mu$.*

Definition 9. For every $f\in M(\mu)$ we put $\int_X f\cdot d\mu=\lim_n\int_X f\cdot 1_{Z_n}d\mu$, where $\{Z_n\}$ is an arbitrary sequence of measurable sets such that $Z_n\nearrow X$ and $f\cdot 1_{Z_n}\in E_0$.

By Proposition 7 there exists a sequence $\{Z_n\}$ such that $\{\int f\cdot 1_{Z_n}d_\mu\}$ is convergent in F. If another sequence $\{W_n\}$ has the same property, then by 4 from Definition 2 we get $X\setminus Z_n\xrightarrow{(\mu)}0$ and $X\setminus W_n\xrightarrow{(\mu)}0$ and using Proposition 8 we obtain $\lim_n\int f\cdot 1_{Z_n}d\mu=\lim_n\int f\cdot 1_{W_n}d\mu$. In conclusion the integral of functions from $M(\mu)$ is well defined.

Theorem 10 *(Absolute Continuity Theorem). If $f\in M(\mu)$ and $A_n\xrightarrow{(\mu)}0$, then $\int f\cdot 1_{A_n}d\mu\to 0$.*

Proof. a) If $B_n\xrightarrow{(\mu)}0$ and $f\cdot 1_{B_n}\in E_0$, then from Def. 5 $\int f\cdot 1_{B_n}d\mu\to 0$.

b) If $D_n\xrightarrow{(\mu)}0$ and $f\cdot 1_{X\setminus D_n}\in E_0$, then by Proposition 8 and Definition 9 $\int f\cdot 1_{X\setminus D_n}d\mu\to\int f\,d\mu$, hence $\int f\cdot 1_{D_n}d\mu\to 0$, since $\int f\,d\mu=\int f\cdot 1_{X\setminus D_n}d\mu$

$+\int f\cdot 1_{D_n} d\mu$. Let us define $B_n=\{x\in A_n;\ \rho(f(x), 0)\leq n\}$, $D_n=A_n\setminus B_n$. Observe, that $\int f\cdot 1_{A_n} d\mu=\int f\cdot 1_{B_n} d\mu+\int f\cdot 1_{D_n} d\mu\to 0$, because $\int f\cdot 1_{B_n} d\mu\to 0$ from a) and $\int f\cdot 1_{D_n} d\mu\to 0$ from b).

Definition 11. For $f_n, f\in M(X, S)$ we define a convergence in measure by the formula

$f_n \xrightarrow{(\mu)} f$ iff for every $\varepsilon>0$ $E_n(\varepsilon)\xrightarrow{(\mu)} 0$, where $E_n(\varepsilon)=\{x\in X;\ \rho(f_n(x), f(x))\geq\varepsilon\}$

Proposition 12. *If $f_n(x)\to f(x)$ for every $x\in X$, then $f_n\xrightarrow{(\mu)} f$.*

Proof. Fix $\varepsilon>0$ and put $A_k=\bigcup_{n=k}^{\infty} E_n(\varepsilon)$, then $\{A_k\}$ is a non-increasing sequence of sets and $\bigcap_{k=1}^{\infty} A_k=\emptyset$. Indeed, $x\in\bigcap_{k=1}^{\infty} A_k$ if and only if for every $k\in N$ there is a $n\geq k$ such that $\rho(f_n(x), f(x))\geq\varepsilon$, while by pointwise convergence we get a $k\in N$ such that for all $n\geq k$ $\rho(f_n(x), f(x))<\varepsilon$. This contradiction implies the fact $A_k\searrow\emptyset$ and consequently $A_k\xrightarrow{(\mu)} 0$. Obviously $E_k\xrightarrow{(\mu)} 0$ because $E_k\subset A_k$.

Theorem 13 *(Convergence Theorem of Lebesgue-Vitali type). If $f_n\xrightarrow{(\mu)} f$ $(f, f_n\in M(\mu))$, there exists a $g\in M(\mu)$ such that for every $x\in X$ and $n\in N$ $\rho(0, f_n(x))\leq\rho(0, g(x))$ and*

$(*)$ $\lim_k \sup_n \|\int f_n\cdot 1_{D_k} d\mu\|=0$, *whenever* $D_k\xrightarrow{(\mu)} 0$, *then* $\int f_n d\mu\to\int f d\mu$.

Proof. Define $Z_m=g^{-1}(B(0, m))$, fix $\eta>0$, observe that $X\setminus Z_m\xrightarrow{(\mu)} 0$. Since $f\in M(\mu)$ and $(*)$ is fulfilled, it follows that for sufficiently large fixed m

i) $\sup_l \|\int f_l\cdot 1_{X\setminus Z_m} d\mu\|<\eta/6$;

ii) $\|\int f\cdot 1_{X\setminus Z_m} d\mu\|<\eta/6$.

Observe that for $x\in Z_m$ we have $\rho(f_n(x), 0)\leq\rho(g(x), 0)\leq m$. Fix $\delta>0$ such that $sv_\delta(\mu_m, X)<\eta/3$ (cf. 3 in Definition 2). Putting $E_n(Z_m, \delta)=\{x\in Z_m;\ \rho(f_n(x), f(x))\geq\delta\}$, we obtain $E_n(Z_m, \delta)\xrightarrow{(\mu)} 0$ as $n\to\infty$, therefore there exists N_1 such that for every $n\geq N_1$;

iii) $\sup_l \|\int f_l\cdot 1_{E_n} d\mu\|<\eta/6$;

iv) $\|\int f\cdot 1_{E_n} d\mu\|<\eta/6$.

Compute $\|\int f_n\cdot d\mu-\int f d\mu\|\leq\|\int f_n\cdot 1_{X\setminus Z_m} d\mu\|+\|\int f\cdot 1_{X\setminus Z_m} d\mu\|+\|\int f_n\cdot 1_{Z_m} d\mu$ $-\int f\cdot 1_{Z_m} d\mu\|\leq\eta/6+\eta/6+\|\int f_n\cdot 1_{E_n} d\mu\|+\|\int f\cdot 1_{E_n} d\mu\|+\|\int f_m\cdot 1_{Z_m\setminus E_n} d\mu$ $-\int f\cdot 1_{Z_m\setminus E_n} d\mu\|\leq(2\eta/3)+sv_\delta(\mu_m, X)<(2\eta/3)+(\eta/3)=\eta$ for $n\geq N_1$.

In the above computation we used inequalities i) — iv) and the fact that $\|\int u\, d\mu-\int v\, d\mu\|\leq sv_\delta(\mu_\alpha, X)$ for $u, v\in E_0$ such that $\sup_x \rho(u(x), v(x))<\delta$ and $\sup_{x\in X}\rho(u(x), 0)<\alpha$, $\sup_{x\in X}\rho(v(x), 0)<\alpha$, which was proved in [6].

Proposition 14. *One can observe that if in assumptions of Theorem 13 we take $f_n(x)\to f(x)$ for all $x\in X$ instead of $f_n\xrightarrow{(\mu)} f$, then by Proposition 12 the Theorem does not cease to hold.*

Examples: 15.1. Let S and F be two Banach spaces, $X=[0, 1]$, $\mathcal{R}$ be a σ-ring of Lebesgue measurable subsets of $[0, 1]$, $\varphi: S\to F$ be a uniformly (locally) continuous function. Put $\mu(A)\ r=\varphi(r)\cdot m(A)$, where m denotes the Lebesgue measure, then μ is an operator measure in the above sense. If $\varphi\circ f: [0, 1]\to F$ is a summable function in the sense of Bochner, then $f\in M(\mu)$ and $\int f\, d\mu=\int(\varphi\circ f) dm$.

In particular, if $S=F=R$ and $\varphi(r)=r$, then we obtain simply a Lebesgue measure on $[0, 1]$.

15.2. Let $(X, \mathscr{R})$ be a measure space, S and F be two Banach spaces and $N(S, F)=L(S, F)$, where $L(S, F)$ is the space of all linear bounded operators equipped with the pointwise convergence topology. Let $\mu:\mathscr{R}\to L(S, F)$ be an operator measure in the sense of Definition 2. In this particular situation for every $f\in M(\mu)$ the Dobrakov integral (cf. 4) is well defined and both integrals are equal.

Conversely every integrable function in the sense of Dobrakov is a member of $M(\mu)$. Since every integrable function in the sense of Bartle (cf. [1]) is in the sense of Dobrakov as well, it follows that it is integrable in the sense of Definition 9.

15.3. Let us consider a measurable space $(X, \mathscr{R}, m)$, assume m to be finite and atomless and put $S=R, F=L^1(X)$. Let $k: X^2\times R\to R$ be a Caratheodory function such that $\int k(\cdot, y, r)dm(y)\to L^1(X)$ for $r\in R$. Defining $\mu(A)r=\int_A k(\cdot, y, r)\,dm(y)$, we obtain an operator measure and for $f\in M(\mu)$ $(\int_X f\cdot d\mu)(x)=\int k(\cdot, x, y, f(y))\,dm(y)$.

15.4. Let X and F be the same as in the previous example and $f: X\times R\to R$ be a Caratheodory function with $f(\cdot, 0)\equiv 0$ and $f(\cdot, r)$ be a member of $L^1(X)$. Defining $\mu(A)r=f(\cdot, 1_A, r)$, we get an operator measure μ and for any $g\in M(\mu)$ we have $(\int_X g\,d\mu)(x)=f(x, g(x))$.

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