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AN IMBEDDING THEOREM FOR SOBOLEV-ORLICZ SPACES VIA INTERPOLATION

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Summary. Let $G \subset R^N$ be a bounded domain of the class $C^{0,1}$. Using a suitable interpolation method we shall show that the Sobolev-Orlicz space $W^1L_M(G)$ is continuously imbedded into the Orlicz space $L_A(G)$ with $A^{-1}(t)=M^{-1}(t)/t^{1/N}$ whenever $M^{-1}(t) \sim t^{1/p_0}h(t^{1/p_1-1/p_0})$ with $1<p_0<p_1<N$ and h concave.

1. Introduction. A function $M: R \to \langle 0, \infty)$ is said to be the Young function, if M is even, convex and $\lim_{t\to 0} M(t)/t = \lim_{t\to\infty} t/M(t)=0$. Let $G\subset R^N$ be a bounded domain of the class $C^{0,1}$ (see, e. g. [7]); i. e. the boundary of G can be locally described by a finite number of Lipschitz-continuous functions in suitable Cartesian coordinate systems.

For any measurable function f on G let us denote $\rho_M(f)=\int_G M(f(x))dx$. If $k \geqq 0$ is an integer, then we define the Sobolev-Orlicz space

$$W^kL_M(G)=\{f;\ \text{there exists } c\neq 0 \text{ such that } \rho_{k,M}(f)=\sum_{|i|\leqq k}\rho_M(cD^if)<\infty\},$$

$i=(i_1, \ldots, i_N)$ beeing multiindices with the length $|i|=i_1+\cdots+i_N$ and D^if being generalized derivatives of f; $W^kL_M(G)$ is equipped with the (Luxemburg) norm $\|f\|_{k,M}=\inf\{c>0: \rho_{k,M}((1/c)f)\leqq 1\}$. For $k=0$ we usually write $L_M(G)$ (the Orlicz space) instead of $W^0L_M(G)$. The theory of Sobolev-Orlicz spaces is contained e. d. in [1, 7], the Orlicz spaces are especially studied in [5].

The great importance of imbedding theorems led several authors [9, 8, 3] to the study of 'Sobolev-Orlicz analogies' of the known and frequently used imbeddings of Sobolev spaces into Lebesgue (or Hölder) spaces. The most complete theory was given in [3]. In the sequel the notation $X \frown Y$ will denote that the space X is continuously imbedded into the space Y. Here we shall deal with imbeddings of the type $W^1L_M(G) \frown L_A(G)$, which correspond to Sobolev imbeddings $W^1_p(G) \frown L_q(G)$ for $p<N$. The restriction $k=1$ can be readily removed. In [3] there is proved the following imbedding theorem: If

$$\int_0^1 M^{-1}(t)/t^{1+1/N}\,dt<\infty,\quad \int_1^\infty M^{-1}(t)/t^{1+1/N}\,dt=\infty,$$

then $W^1L_M(G) \frown L_A(G)$, where $A^{-1}(t)=\int_0^t M^{-1}(u)/u^{1+1/N}\,du$.

(Cf. also [9].) Both proofs in [9] and [3] are rather complicated and lengthy. In the next section we shall give a very short proof, showing the power of the interpolation technics.

2. An Imbedding Theorem. Our main tool will be the interpolation method for Orlicz spaces developed in [4] and we shall briefly show the possibility of its extension to the Sobolev-Orlicz spaces. We shall need the following special interpolation result from [4]: Let $1<p_0<p_1<\infty$ and let h be a positive concave function on $\langle 0, \infty)$, $h(0)=0$. Let M be a Young function such that

$$M^{-1}(t) \sim t^{1/p_0} h(t^{1/p_1-1/p_0}). \tag{1}$$

(The notation $f\sim g$ means that $cf\leq g\leq Cf$ for some constants c, $C>0$.) Then $L_M(G)$ is an interpolation space with respect to the couple $\{L_{p_0}(G), L_{p_1}(G)\}$. Moreover, the mapping $\{L_{p_0}(G), L_{p_1}(G)\}\to L_M(G)$ is an interpolation functor. (For the general interpolation theory see e. g. [2, 10].)

It is easy to prove that for the validity of (1) it is sufficient to suppose that

$$M(st)\leq C \max(s^{p_0}, s^{p_1})M(t), \quad s, t>0, \tag{2}$$

which is, in fact, a sharp form of the so-called Δ_2-condition; it expresses that M is 'between two powers'.

This interpolation procedure can be carried over to Sobolev-Orlicz spaces on bounded sufficiently smooth domains. One needs to define potential Sobolev-Orlicz spaces with the aid of Bessel potentials, further the corresponding extension theorem (from G to R^N) and a multiplier theorem of the Michlin type. This is done for another interpolation method in Orlicz spaces in [6] and it works also here without essential changes. So we get that the space $W^1L_M(G)$ with M given by (1), $1<p_0<p_1<\infty$, is an interpolation space with respect to $\{W^1_{p_0}(G), W^1_{p_1}(G)\}$ and the mapping $\{W^1_{p_0}(G), W^1_{p_1}(G)\}\to W^1L_M(G)$ is an interpolation functor.

Theorem. *Let M be a Young function satisfying* (2) *with some* $1<p_0<p_1<\infty$. *Then* $W^1L_M(G)\frown L_A(G)$, *where* $A^{-1}(t)=M^{-1}(t)/t^{1/N}$.

Proof. By the Sobolev imbedding theorem ([1, 2, 7, 10]) $W^1_{p_i}(G) \frown L_{q_i}(G)$, $1/q_i=1/p_i-1/N$, $i=0, 1$. We get at once $W^1L_M(G)\frown L_A(G)$, where $M^{-1}(t)\sim t^{1/p_0}h(t^{1/p_1-1/p_0})$ and $A^{-1}(t)\sim t^{1/q_0}h(t^{1/q_1-1/q_0})$. This yields

$$A^{-1}(t)\sim t^{1/p_0-1/N}h(t^{1/p_1-1/p_0})=M^{-1}(t)/t^{1/N}.$$

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