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ON UNICITY AND STRONG UNICITY OF BEST APPROXIMATION IN THE L_1-NORM

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Summary. Let X be a normed linear space and consider a finite dimensional subspace M in X. Then for any $x \in X$ there exists a best approximation g^* out of M. The subspace M is called a unicity subspace, if each $x \in X$ possesses a unique best approximation in M. In the case, when M is a unicity subspace and 0 is the best approximation of $x \in X$, a question of practical interest is that of how fast the 'nearly best approximations' $g \in M$, satisfying $\|x-g\| \leqq \|x\|+\delta$, approach zero as $\delta \to 0$. This is the problem of strong unicity of best approximation. In the present paper we give a survey of some recent results on unicity and strong unicity of best approximation in the L_1-norm and discuss the application of these results.

Let X be a normed linear space and consider a finite dimensional subspace M in X. Then each $x \in X$ possesses a best approximation g^* out of M, i. e.

$$\|x-g^*\| = \inf\{\|x-g\| : g \in M\}.$$

If 0 is a best approximation of x, we say that x is orthogonal to M, written $x \perp M$. Furthermore, the subspace M is called a 'unicity' subspace of X, if each $x \in X$ possesses a unique best approximation in M. The study of unicity subspaces is an important problem in approximation theory.

In the case when M is a unicity subspace of X, a more delicate question consists in investigation of the strong unicity of best approximation. Let $x \in X$ be orthogonal to M and set

$$\Omega(x, \delta)_X = \sup\{\|g\| : \|x-g\| \leqq \|x\|+\delta,\ g \in M\} \quad (\delta>0).$$

This is the so-called modulus of strong unicity. If M is a unicity subspace of X, then, clearly, $\Omega(x, \delta)_X$ tends to zero as $\delta \to 0$ for any $x \in X$ orthogonal to M. The modulus of strong unicity shows how fast the 'nearly best approximations', satisfying $\|x-g\| \leqq \|x\|+\delta$, approach zero, when $\delta \to 0$.

In case of Chebyshev approximation the unicity subspaces were studied by many authors, e. g. Haar [6], Kolmogorov [8] and others. The strong unicity of Chebyshev approximation was investigated by Newman and Shapiro [17] (real and complex functions) and Wegmann [21] (vector valued functions).

In the present paper we shall give a survey of some recent results on unicity and strong unicity of best approximation in the L_1-norm. We shall also discuss the application of these results.

By a well-known result of Krein [9] the space L_1 does not contain finite dimensional unicity subspaces. Therefore we consider the space $X=C_1$ of real valued continuous functions on [0, 1] with norm $\|f\|_1=\int_0^1|f(x)|\,dx$ (Lebesgue integral). Further, let M be a finite dimensional subspace of C_1. Recall, that M is said to be a Haar subspace, if zero is the only element in M, possessing more than dim $M-1$ zeros on [0,1].

The classical result of Jackson [7] and Krein [9] states that any Haar subspace of C_1 is a unicity subspace. The strong unicity of L_1-approximation for Haar subspaces was studied by Björnestál [1, 2]. (Some partial results are implicitly contained in Usow [20].) Set

$$\omega_f(h)_\gamma=(\omega_f^{-1}(h)\,h^\gamma)^{-1}\quad(\gamma>0), \tag{1}$$

where ω_f denotes the modulus of continuity of $f\in C_1$ and -1 is used to denote the inverse functions. ($\omega_f(h)=\sup\{|f(x_1)-f(x_2)|:x_1, x_2\in[0, 1], |x_1-x_2|\leq h\}$. Evidently, $\omega_f(h)_\gamma$ is also a modulus of continuity. In particular, if $\omega_f(h)=O(h^\alpha)$ then $\omega_f(h)_\gamma=O(h^{\alpha/(\alpha\gamma+1)})$ $(0<\alpha\leq1)$.

In [1] the following result is proved

Theorem A (Björnestál). *Let $X=C_1$ and let M be a Haar subspace of C_1. Then for any $f\in C_1$ orthogonal to M*

$$\Omega(f, \delta)_{C_1}\leq c_{f,M}\,\omega_f(\delta)_1\quad(0<\delta\leq1), \tag{2}$$

where the constant $c_{f,M}$ depends on f and M. Moreover this estimation is in general the best possible.

Later in [12] it was shown that the constant in (2) can be chosen depending only on ω_f and M.

The Björnestál's result gave the solution of the strong unicity problem in case, when M is a Haar subspace. But in contrast with Chebyshev approximation the Haar subspaces are not the only unicity subspaces of C_1. For example Galkin [5] and Strauss [19] proved that the subspace of splines with fixed knots is also a unicity subspace in C_1. Some similar results for another families of spline functions were given by Carroll and Braess [3] and Sommer [18]. A constructive characterization of unicity subspaces of C_1 is still unknown. Nevertheless the rate of strong unicity of L_1-approximation can be determined for arbitrary unicity subspaces of C_1. We proved the following

Theorem B ([15]). *Let $X=C_1$ and let M be a unicity subspace of C_1. Then for any $f\in C_1$ such that $f\perp M$*

$$\Omega(f, \delta)_{C_1}\leq c_{f,M}\,\omega_f(\delta)_1\quad(0<\delta\leq1)$$

and this estimation is sharp in general.

Thus it turned out that Björnestál's result can be extended to any unicity subspaces. In particular, it holds for different families of spline functions, too. This extension was based on some delicate properties of unicity subspaces of C_1 given by Cheney and Wulbert [4].

In 1965 Kripke and Rivlin [11] generalized the Jackson-Krein unicity theorem for complex valued functions. Denote by $\overline{C_1}$ the space of complex valued continuous functions on [0, 1] endowed with the L_1-norm and let M be a finite dimensional subspace of $\overline{C}_1$. The definition of Haar subspace of $\overline{C}_1$ is analogous to the real case.

Theorem C (Kripke-Rivlin). *Let $X=\overline{C}_1$ and let M be a Haar subspace of $\overline{C}_1$, possessing a real basis. Then M is a unicity subspace of $\overline{C}_1$.*

In [11] it is also shown that unicity may fail, if M does not possess a real basis.

This result of Kripke and Rivlin raises the question of strong unicity of complex L_1-approximation. In [14] we investigated the unicity and strong unicity of L_1-approximation in a more general situation for vector valued functions. Let $C_1^{(m)}$ be the space of continuous functions on [0, 1], taking values in $\mathbf{R}^m$ $(m\geq 1)$, and endowed with norm $\|f\|_m=\int_0^1 |f(x)|_m\,dx$, where $f=(f_1, f_2, \ldots, f_m)$, $|f(x)|_m=(\Sigma_{i=1}^m f_i^2(x))^{1/2}$. The modulus of continuity of $f\in C_1^{(m)}$ is given by $\omega_f(h)=\sup\{|f(x_1)-f(x_2)|_m : |x_1-x_2|\leq h,\ x_1, x_2\in[0, 1]\}$, $\omega_f(h)_\gamma$ is defined as in (1). Furthermore, let H_i $(1\leq i\leq m)$ be arbitrary Haar subspaces of C_1 $(C_1=C_1^{(1)})$ and set $M=H_1\times H_2\times\ldots\times H_m$, i. e. the subspace $M\subset C_1^{(m)}$ is Cartesian product of Haar subspaces of C_1. Then we have the following

Theorem D ([14]). *Let $X=C_1^{(m)}$ $(m\in\mathbf{N})$ and let $M\subset C_1^{(m)}$ be a Cartesian product of Haar subspaces of C_1. Then for any $f\in C_1^{(m)}$ such that $f\perp M$*

$$\Omega(f, \delta)_{C_1^{(m)}}\leq c_{f, M}\,\omega_f(\delta)_{\gamma(m)}, \tag{3}$$

where $\gamma(1)=1$ and $\gamma(m)=2$, if $m\geq 2$.

By this theorem we easily obtain a result on unicity of L_1-approximation for vector valued functions.

Corollary ([14]). *Let M be a Cartesian product of m Haar subspaces of C_1. Then M is a unicity subspace of $C_1^{(m)}$ $(m\in\mathbf{N})$.*

This corollary is an extension of Jackson-Krein theorem for vector valued functions. Moreover, for $m=2$ we obtain a statement, which is even more general than the theorem of Kripke and Rivlin. Indeed, in Theorem C the assumption that $M\subset\overline{C}_1$ is a Haar subspace, possessing a real basis, is equivalent to the fact that M is a Cartesian product of two equal Haar subspaces of C_1. This restriction is omited in our theorem, since we consider Cartesian products of arbitrary Haar subspaces of C_1. Furthermore, for $m=1$ (3) gives the same estimation as (2), i. e. we obtain Björnestál's result. But the question of sharpness of estimation (3) for $m\geq 2$ remains open. It seems to the author that (3) cannot be improved in general.

The strong unicity type results can be widely applied in the study of the rate of convergence of computational algorithms.

The usual approach to the solution of L_1-approximation problem consists in replacing the L_1-norm by the discrete L_1-norm. The discrete L_1-norm of $f\in C_1^{(m)}$ is given by

$$\|f\|_m^{(k)}=\sum_{i=0}^{k-1} k^{-1}|f(i/k)|_m \quad (m\geq 1,\ k\geq 1). \tag{4}$$

The best approximation in norm (4) can be obtained as a solution of linear programming problem. A question of practical interest here is the investigation of behaviour of best discrete L_1-approximants as $k\to\infty$. The convergences of discrete L_1-approximants to the best L_1-approximation as $k\to\infty$ wa established by Motzkin and Walsh [16]. (In a more general setting this was done by Kripke [10].) The results on strong unicity of L_1-approximation can be used in estimating the rate of convergence of discrete L_1-approximants. The first partial results in this direction were given by Usow [20]. A further discussion of this problem can be found in [13], where sharp estimates are already given. In [20] and [13] the convergence of discrete L_1-approximants was studied in the case when functions are real and the approximating subspace satisfies the Haar property. Applying Theorem B, these investigations can be extended to the case of arbitrary unicity subspaces of C_1. Thus, in particular, we can obtain estimations for the rate of convergence of discrete L_1-approximants, when approximating subspaces are different families of spline functions. Moreover, using Theorem D, we can extend these investigations to the vector valued functions.

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