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SOME IMBEDDINGS FOR WEIGHTED SOBOLEV SPACES

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Summary. The paper deals with weighted L^p-estimates of functions by certain weighted L^p-norms of the first derivatives; these estimates, derived for smooth functions, can be used for to derive imbedding assertions for Sobolev weight spaces. Two types of weight functions are considered: general weight functions and special weight functions depending on the distance from the boundary.

0. Introduction. 0.1. Let Ω be a domain in $\boldsymbol{R}^N$ and let $a_0, a_1, \ldots, a_N$ be functions defined on Ω, positive a. e. on Ω and such that $a_i \in L^1_{\text{loc}}(\Omega)$, $a_i^{-1/(p-1)}$ $L^1_{\text{loc}}(\Omega)$ for $i=0, 1, \ldots, N$ with $p>1$. The functions a_i are called *weight functions*; we shall denote by a the vector function $a=\{a_1, \ldots, a_N\}$.

Further, let $L^p(\Omega; a_0)$ be the set of all functions $u=u(x)$ such that

$$\| u \|_{p;a_0} = \left(\int_\Omega | u(x) |^p a_0(x)\, dx\right)^{1/p} < \infty. \tag{0.1}$$

Let us denote by $\| \cdot \|_{1,p;a}$ the expression defined for u by

$$\| u \|^p_{1,p;a} = \Sigma_{i=1}^N \| \partial u / \partial x_i \|^p_{p;a_i}.$$

0.2. *Sobolev weight spaces.* We shall denote by

$$W^{1,p}(\Omega; a_0, a)$$

the closure of the set of all functions $u \in C^1(\overline{\Omega})$ such that

$$\| u \|_{p;a_0} + \| u \|_{1,p;a} \tag{0.2}$$

is finite, the closure being taken with respect to the norm (0.2).

Further, we shall denote by $W^{1,p}_0(\Omega; a_0, a)$ the closure of the set $C_0^\infty(\Omega)$ with respect to the norm (0.2).

Obviously, the sets $L^p(\Omega; a_0)$ with the norm (0.1) and $W^{1,p}(\Omega; a_0, a)$, $W^{1,p}_0(\Omega; a_0, a)$ with the norm (0.2) are Banach spaces.

0.3. The aim of this paper is to derive inequalities of the type

$$\int_\Omega | u(x) |^p b_0(x)\, dx \leq C \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i}(x) \right|^p a_i(x)\, dx, \tag{0.3}$$

i. e.

$$\| u \|_{p;b_0} \leq C^* \| u \|_{1,p;a}. \tag{0.4}$$

for smooth functions u with constants C, C^* independent of u. More precisely, we are interested in determining for *what* weight functions b_0 the above estimate holds.

Inequalities of the form (0.3) enables us to derive imbeddings as

$$W^{1,p}(\Omega; a_0, a) \nearrow L^p(\Omega; b_0) \tag{0.5}$$

and

$$W_0^{1,p}(\Omega; a_0, a) \nearrow L^p(\Omega; b_0). \tag{0.6}$$

We shall also deal with a certain modification of (0.3).

0.4. The domain. We shall suppose that Ω is a bounded domain with a boundary $\partial\Omega$ which is — roughly speaking — Lipschitz-continuous (for a more precise description see [10] or [7]). For such domains the notion of the unit vector ν of the outer normal to $\partial\Omega$ is meaningful a. e. on $\partial\Omega$, $\nu = (\nu_1, \ldots, \nu_N)$.

1. General Weight Functions. 1.1. First of all we shall deal with the inequality (0.3) for the case $p=2$. The result is summarized in the following theorem, the proof of which is based on ideas of Beesack [1] and Benson [2].

1.2. Theorem. *Let the weight functions a_i belong to $C^1(\Omega)$ for $i=1, \ldots, N$. Further, let there exist a function v such that*

$$b_0(x) = -\left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial v}{\partial x_1}(x)\right)\right)/v(x) \tag{1.1}$$

is a weight function and that

$$a_i \frac{\partial v}{\partial x_i} / v \in C^1(\bar{\Omega}) \text{ for } i=1, \ldots, N. \tag{1.2}$$

If $u \in C^1(\bar{\Omega})$ is such that

$$u^2 \sum_{i=1}^{N} \left(\frac{\partial v}{\partial x_i} / v_i\right) a_i \nu_i \geq 0 \quad \text{a. e. on } \partial\Omega, \tag{1.3}$$

then the following inequality is valid

$$\int_\Omega |u(x)|^2 b_0(x)\,dx \leq \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^2 a_i(x)\,dx. \tag{1.4}$$

1.3. Remarks. (i) Condition (1.1) can be rewritten as

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial v}{\partial x_i}(x)\right) + b_0(x)v(x) = 0 \quad \text{on } \Omega. \tag{1.5}$$

Therefore, one can say that the inequality (1.4) holds if the weight functions $a_1, \ldots, a_N, b_0$ are such that there exists a solution v of the partial differential equation (1.5).

(ii) Condition (1.2) can be weakened in various directions. E. g., one can replace (1.2) by the condition

$$u^2\left(\frac{\partial v}{\partial x_i} / v\right) a_i \in W^{1,1}(\Omega), \quad i=1, \ldots, N.$$

1.4. Proof of Theorem 1.2. Obviously,

$$0 \leq \sum_{i=1}^{N} [\frac{\partial u}{\partial x_i} a_i^{1/2} - u (\frac{\partial v}{\partial x_i} / v) a_i^{1/2}]^2$$

$$= \sum_{i=1}^{N} [(\frac{\partial u}{\partial x_i})^2 a_i + u^2 (\frac{\partial v}{\partial x_i} / v)^2 a_i - 2u \frac{\partial u}{\partial x_i} (\frac{\partial v}{\partial x_i} / v) a_i].$$

This together with the identity

$$2u \frac{\partial u}{\partial x_i} (\frac{\partial v}{\partial x_i} / v) a_i = \frac{\partial}{\partial x_i} (u^2 (\frac{\partial v}{\partial x_i} / v) a_i) - u^2 \frac{\partial}{\partial x_i} ((\frac{\partial v}{\partial x_i} / v) a_i)$$

implies the inequality

$$\sum_{i=1}^{N} [(\frac{\partial u}{\partial x_i})^2 a_i + u^2 (\frac{\partial v}{\partial x_i} / v)^2 a_i + u^2 \frac{\partial}{\partial x_i} ((\frac{\partial v}{\partial x_i} / v) a_i)] \tag{1.6}$$

$$\geq \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (u^2 (\frac{\partial v}{\partial x_i} / v) a_i).$$

Since the identity

$$\sum_{i=1}^{N} [(\frac{\partial v}{\partial x_i} / v)^2 a_i + \frac{\partial}{\partial x_i} (a_i \frac{\partial v}{\partial x_i} / v)] = -b_0 \tag{1.7}$$

is obvious, we obtain from (1.6) and (1.7) the inequality

$$\sum_{i=1}^{N} (\frac{\partial u}{\partial x_1})^2 a_i (x) - u^2 b_0 (x) \geq \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (u^2 (\frac{\partial v}{\partial x_i} / v) a_i).$$

Integration over Ω and Green's formula yield

$$\int_{\Omega} [\sum_{i=1}^{N} (\frac{\partial u}{\partial x_i})^2 a_i (x) - u^2 b_0 (x)] \, dx \geq \int_{\partial\Omega} u^2 [\sum_{i=1}^{N} (\frac{\partial v}{\partial x_i} / v) a_i \nu_i] \, dS,$$

which implies the inequality (1.4) in view of condition (1.3).

1.5. Examples. Let $N=2$, $\Omega=(0,1)\times(0,1)$.

(i) Let $a_i(x)=x_i^2$, $i=1, 2$. Taking $v(x)=(x_1 x_2)^{-1/2}$, we obtain by (1.1) $b_0(x)=1/2$. An investigation of condition (1.3) shows that we have to consider functions $u \in C^1(\overline{\Omega})$ such that $u(x)=0$ for $x \in \partial\Omega$ with either $x_1=1$, or $x_2=1$. For such functions u we have the inequality

$$\int_{\Omega} |u(x)|^2 \, dx \leq 2[\int_{\Omega} (\frac{\partial u}{\partial x_1})^2 x_1^2 dx + \int_{\Omega} (\frac{\partial u}{\partial x_2})^2 x_2^2 \, dx]. \tag{1.8}$$

In particular, this inequality holds for $u \in C_0^1(\Omega)$.

(ii) Let $a_i(x)=(x_1^2+x_2^2)^2/x_i$, $i=1, 2$. Taking $v(x)=(x_1^2+x_2^2)^{-1/2}$, we obtain $b_0(x)=x_1+x_2$. Now (1.3) implies that for $u \in C^1(\overline{\Omega})$ such that $u(x)=0$ for $x \in \partial\Omega$ with either $x_1=0$ or $x_2=0$ (and in particular for $u \in C_0^1(\Omega)$) we have

$$\int_{\Omega} |u(x)|^2 (x_1+x_2) \, dx \leq \sum_{i=1}^{2} \int_{\Omega} (\partial u/\partial x_i)^2 x_i^{-1} (x_1^2+x_2^2)^2 \, dx. \tag{1.9}$$

(iii) Let $a_i(x)=x_1^2+x_2^2$, $i=1, 2$. Taking $v(x)=(x_1^2+x_2^2)^{-1}$, we obtain $b_0(x)=4(x_1^2+x_2^2)$. So, for the same functions u as in (i) we have the inequality

$$\int_\Omega |u(x)|^2(x_1^2+x_2^2)\,dx \leq \frac{1}{4} \sum_{i=1}^{2} \int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 (x_1^2+x_2^2)\,dx. \tag{1.10}$$

1.6. Remarks. (i) It follows easily that the inequalities (1.8) etc. hold for a general domain $\Omega\subset \boldsymbol{R}^2$, if we suppose $u\in C_0^1(\Omega)$. Especially, such domains are of interest that contain parts of the coordinate axes, on which the weight functions a_i degenerate. (Naturally, one has to make some minor changes; e. g. in 1.5 (i) it is necessary to take $v(x)=(x_1^2+x_2^2)^{-1/4}$ in order to avoid problems with negative x_i's.)

(ii) Inequalities (1.8) and (1.10) are closely connected; it is obvious that a 'worse' weight function $b_0(x)$ is eliminated by a 'better' constant C in an estimate of the type (0.3).

(iii) The estimates derived in the above examples can be obtained by a repeated or modified use of the one-dimensional Hardy inequality (see [3, Theorem 330]); the constants obtained in these examples are in some cases better.

There is an analogue of Theorem 1.2 for general $p>1$. We give here the corresponding assertion without proof:

1.7. Theorem. *Let $p>1$. Let the weight functions a_i belong to $C^1(\Omega)$ for $i=1, \ldots, N$. Further, let there exist a function v such that $v\geq 0$ and $\partial v/\partial x_i \geq 0$ on Ω $(i=1, \ldots, N)$,*

$$b_0(x) = -\left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\left(\frac{\partial v}{\partial x_i}(x)\right)^{p-1}\right)\right)/v^{p-1}(x)$$

is a weight function and $a_i\left(\frac{\partial v}{\partial x_i}/v\right)^{p-1}\in C^1(\overline{\Omega})$ for $i=1, \ldots, N$.

If $u\in C^1(\overline{\Omega})$ is such that

$$|u|^p \sum_{i=1}^{N} \left(\frac{\partial v}{\partial x_i}/v\right)^{p-1} a_i \nu_i = 0 \text{ a. e. on } \partial\Omega,$$

then the following inequality holds:

$$\int_\Omega |u(x)|^p\, b_0(x)\,dx \leq \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^p a_i(x)\,dx.$$

1.8. Remarks. (i) Analogous remarks as in Section 1.3 can be made again for Theorem 1.7.

(ii) If the conditions $\partial v/\partial x_i \geq 0$ $(i=1, \ldots, N)$ are replaced by conditions $\partial v/\partial x_i \leq 0$ $(i=1, \ldots, N)$, an analogous assertion as in Theorem 1.7 holds with $-\partial v/\partial x_i$ instead of $\partial v/\partial x_i$ and with $-b_0(x)$ instead of $b_0(x)$. (The condition $v\geq 0$ remains unchanged.)

(iii) For $p=2, 4, 6, \ldots$ the conditions $v\geq 0$, $\partial v/\partial x_i \geq 0$ (or $\partial v/\partial x_i \leq 0$) can be omitted.

2. Special Weight Functions. 2.1. In this part we shall be concerned with weight functions a_i of the type

$$a_i(x)=s_i(\operatorname{dist}(x, M)), \tag{2.1}$$

where M is a certain part of the boundary $\partial\Omega$ of the domain Ω and $s_i=s_i(t)$ are positive continuous functions defined for $t>0$. Further, we shall suppose that

$$s_1(t)=s_2(t)=\ldots=s_N(t)=s(t) \tag{2.2}$$

and that the weight function b_0 is given in terms of a positive continuous function $\sigma_0=\sigma_0(t)$ by the formula $b_0(x)=\sigma_0(\operatorname{dist}(x, M))$.

2.2. For these special weight functions, inequalities of the type (0.3) appear in the literature: So it can be shown that for $s(t)=t^\varepsilon$ we have (under certain assumptions) $\sigma_0(t)=t^{\varepsilon-p}$ (see e. g. [10] or [6]), and for more general $s(t)$ one can take $\sigma_0(t)=s^{-1/(p-1)}(t)[\int_0^t s^{-1/(p-1)}(\tau)\,d\tau]^{-p}$ (see e. g. [6]). It is the Hardy inequality which plays here an important role — either in its classical form (see [3, Theorem 330]) or in a certain generalized form (see e. g. [8]). Here we shall use the following generalization which covers all the cases mentioned above.

2.3. Generalized Hardy inequality. The necessary and sufficient condition for the inequality

$$\Big(\int_0^\infty |f(t)|^q \sigma_0(t)\,dt\Big)^{1/q} \leq C\Big(\int_0^\infty |f'(t)|^p s(t)\,dt\Big)^{1/p} \tag{2.3}$$

to be valid with $1<p\leq q<\infty$ and with a constant C independent of f is

(i) the condition

$$\sup_{t>0}\Big(\int_t^\infty \sigma_0(\tau)\,d\tau\Big)^{1/q}\Big(\int_0^t s^{-1/(p-1)}(\tau)\,d\tau\Big)^{(p-1)/p}<\infty \tag{2.4}$$

for $f(0)=0$ and

(ii) the condition

$$\sup_{t>0}\Big(\int_0^t \sigma_0(\tau)\,d\tau\Big)^{1/q}\Big(\int_t^\infty s^{-1/(p-1)}(\tau)\,d\tau\Big)^{(p-1)/p}<\infty \tag{2.5}$$

for $f(\infty)=0$.

For the proof see [9]; see [4], too. The symbols $f(0)$, $f(\infty)$ stand for the limits of the function $f=f(t)$ continuously differentiable on $(0, \infty)$ for $t\to 0+$ and $t\to\infty$, respectively.

2.4. The domain. Here we shall deal with special domains $G\subset \boldsymbol{R}^N$. We shall suppose that provided $\bar\Delta$ is the closure of the unit cube Δ in $\boldsymbol{R}^{N-1}$, a function $\varphi=\varphi(x')$, $x'=(x_1,\ldots,x_{N-1})$, of the class $C^{0,1}(\bar\Delta)$ is given and

$$G=\{x=(x', x_N);\ x'\in\Delta,\ \varphi(x')-\delta<x_N<\varphi(x')\} \tag{2.6}$$

with a certain fixed $\delta>0$.

The set $M\subset\partial G$ which appears in (2.1) is given by $M=\{x=(x', x_N);\ x'\in\Delta, x_N=\varphi(x')\}$.

2.5. Sobolev weight spaces. Having modified the domain to the form G from Section 2.4, we shall modify the spaces introduced in Section 0.2.

(i) For G from (2.6) we have

$$\int_G |w(x)|^p a_0(x)\,dx=\int_\Delta\Big(\int_{\varphi(x')-\delta}^{\varphi(x')} |w(x', x_N)|^p a_0(x', x_N)\,dx_N\Big)dx'$$

$$= \int_\Delta \left(\int_0^\delta | w(x', \varphi(x') - t) |^p a_0 (x', \varphi(x') - t)\, dt \right) dx'$$

(we have used the substitution $x_N = \varphi(x') - t$ in the inner integral). In what follows we shall work with the so-called spaces with mixed norms: For $p > 1$, $q > 1$ we denote

$$\| w \|_{(p,q);a_0} = \left[\int_\Delta \left(\int_{\varphi(x')-\delta}^{\varphi(x')} | w(x', x_N) |^p a_0 (x', x_N)\, dx_N \right)^{q/p} dx' \right]^{1/q}$$

$$= \left[\int_\Delta \left(\int_0^\delta | w(x', \varphi(x') - t) |^p a_0 (x', \varphi(x') - t)\, dt \right)^{q/p} dx' \right]^{1/q}.$$

The Banach space of functions $w = w(x)$ such that the norm $\| w \|_{(p,q); a_0}$ is finite, will be denoted by $L^{(p,q)}(G; a_0)$. Obviously $L^p(G; a_0) = L^{(p,p)}(G; a_0)$.

(ii) In the definition of Sobolev weight spaces in Section 0.2 we supposed that $u \in L^p(\Omega; a_0)$ and $\partial u / \partial x_i \in L^p(\Omega; a_i)$. If we replace Ω by G from (2.6) and replace the second assumption by $\partial u / \partial x_i \in L^{(p,q)}(G; a_i)$, $i = 1, \ldots, N$, we obtain spaces, which we shall denote by

(2.7) $$W^{1,(p,q)}(G; a_0, a) \text{ and } W_0^{1,(p,q)}(G; a_0, a).$$

(iii) Moreover, we shall suppose that

(2.8) $$\operatorname{supp} u \cap (\partial G - M) = \emptyset.$$

Consequently, a function $u \in W^{1,(p,q)}(G; a_0, a)$ vanishes on ∂G with exception of the part M, and $u \in W_0^{1,(p,q)}(G; a_0, a)$ vanishes on M as well.

2.6. Let us now consider the domain G and the set M from Section 2.4 and the weight functions $s(\operatorname{dist}(x, M))$ and $\sigma_0(\operatorname{dist}(x, M))$. Together with the distance $\operatorname{dist}(x, M)$ one can consider the 'distance of a point $x = (x', x_N) \in G$ to M in the x_N-direction', given by the number $\varphi(x') - x_N$. Since $\varphi \in C^{0,1}(\overline{\Delta})$, both distances are equivalent, i. e., there exists a $c_1 > 0$ such that

(2.9) $$c_1 [\varphi(x') - x_N] \leq \operatorname{dist}(x, M) \leq \varphi(x') - x_N$$

(see [5, Lemma 1.3]).

2.7. Property (H). We say that a continuous positive function $h = h(t)$ defined for $t > 0$ has property (H), if for every pair of positive constants c_1, c_2 there exists a pair of positive constants C_1, C_2 such that $c_1 \leq t/\tau \leq c_2$ implies $C_1 \leq h(t)/h(\tau) \leq C_2$.

If a weight function s from (2.1), (2.2) has property (H), we can in view of inequalities (2.9) consider the weight function $s(\varphi(x') - x_N)$ instead of $s(\operatorname{dist}(x, M))$ and vice versa, and similarly for σ_0.

The main result of this chapter is

2.8. Theorem. *Let us consider the domain G from Section 2.4. Let the functions σ_0, s and the numbers p, q be such that (2.5) is satisfied. Let the functions σ_0, s have property (H). Then the inequality*

(2.10) $$\| u \|_{q;\sigma_0} \leq C \left\| \frac{\partial u}{\partial x_N} \right\|_{(p,q);a}$$

holds for $u \in W^{1,(p,q)}(G; a_0, a)$ with a_i given by (2.1), (2.2) and with a constant $C > 0$ independent of u.

Inequality (2.10) holds for $u \in W_0^{1,(p,q)}(G; a_0, a)$ as well, if σ_0, s, p, q are such that (2.4) is satisfied.

2.9. We omit the proof of Theorem 2.8, which is rather technical: In

$$\| u \|_{q;\sigma_0} = [\int_\Delta (\int_0^\delta | u(x', \varphi(x') - t) |^q \sigma_0(\operatorname{dist}(x, M))\, dt)\, dx']^{1/q}$$

we replace the function $\sigma_0(\operatorname{dist}(x, M))$ in the inner integral by $\sigma_0(t)$ (as a consequence of the fact that σ_0 has property (H)) write $\int_0^\infty$ instead of $\int_0^\delta$ (in view of condition (2.8)), estimate the inner integral using (2.3) for $f(t) = u(x', \varphi(x') - t)$ and go back to the mixed norm $\| \partial u / \partial x_N \|_{(p,q);a}$ with $a(x) = s(\operatorname{dist}(x, M))$.

2.10. Remark. Inequality (2.10) enables us to derive imbeddings of the types (0.5), (0.6), but with the spaces (2.7) on the left-hand sides and with $L^q(G; b_0)$ instead of $L^p(\Omega; b_0)$ on the right-hand sides, $q \geq p$.

2.11. Example. Let us take

$$s(t) = t^\varepsilon, \quad \sigma_0(t) = t^\eta. \tag{2.11}$$

Then conditions (2.6), (2.7) are satisfied for

$$\eta = ((\varepsilon - p + 1) q/p) - 1 \tag{2.12}$$

with $\varepsilon > p - 1$ if $f(\infty) = 0$; $\varepsilon < p - 1$ if $f(0) = 0$.

Since the functions s, σ_0 from (2.11) have property (H), Theorem 2.8 yields the following assertion:

Let $1 < p \leq q < \infty$ and $\varepsilon > p - 1$. Then the inequality (2.10) holds with weight functions defined by (2.11) and η given by (2.12) for $u \in W^{1,(p,q)}(G; a_0, a)$. If $u \in W_0^{1,(p,q)}(G; a_0, a)$, then this inequality holds not only for $\varepsilon > p - 1$, but also for $\varepsilon < p - 1$.

If in particular $q = p$, then it follows from (2.12) that $\eta = \varepsilon - p$ and the result coincides with known results (see Section 2.2).

2.12. The case $\varphi \in C^{0,\lambda}(\bar{\Delta})$, $0 < \lambda < 1$. Let us again consider the domain G from (2.6), but with a function φ, which is only λ-Hölder-continuous. In this case the inequalities (2.9) have to be replaced by $c_1 [\varphi(x') - x_N]^{1/\lambda} \leq \operatorname{dist}(x, M) \leq \varphi(x') - x_N$ (see again Lemma 1.3 in [5]). Using again the method of proof of Theorem 2.8, we obtain — using some monotonicity properties of σ_0, s instead of property (H) — again an inequality of the type (2.10) in a little modified form. We shall not give an exact formulation of the result mentioned; the following example can elucidate the situation:

Let us introduce s and σ_0 by (2.11). If we suppose

$$\varepsilon > \lambda(p-1)$$

and define η by the formula

$$\eta = \begin{cases} p^{-1} [\varepsilon - \lambda(p-1)]\, q - \lambda, & \text{if } \varepsilon \leq \lambda(p-1) + \lambda p/q, \\ (\lambda p)^{-1} [\varepsilon - \lambda(p-1)]\, q - 1, & \text{if } \varepsilon \geq \lambda(p-1) + \lambda p/q, \end{cases}$$

then inequality (2.10) holds with $1 < p \leq q < \infty$ for $u \in W^{1,(p,q)}(G; a_0, a)$.

For $u \in W_0^{1,(p,q)}(G; a_0, a)$ inequality (2.10) holds not only in the abovementioned case, but also for $\varepsilon < \lambda(p-1)$ and for η given by

$$\eta = \begin{cases} p^{-1} [\varepsilon - \lambda(p-1)]\, q - \lambda, & \text{if } \varepsilon \geq 0, \\ [p^{-1}(\varepsilon - p + 1)\, q - 1]\lambda, & \text{if } \varepsilon \leq 0. \end{cases}$$

These results coincide for $q = p$ with the results derived in [5].

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