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A PROBLEM OF k-STRICT CONVEXITY OF FENCHEL-ORLICZ SPACES

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Summary. In the paper a problem of extended k-strict convexity of the Fenchel-Orlicz spaces $L^{\Phi}(\mu, X)$ over not purely atomic measure space (Ω, Σ, μ) with the Luxemburg type norm is studied. The result obtained here shows that the only k-strictly convex Fenchel-Orlicz spaces, where $1 \leq k \leq \infty$, are 1- or ∞-strictly convex spaces. Necessary and sufficient conditions for $L^{\Phi}(\mu, X)$ to be k-strictly convex are also given.

1. Introduction. It is a well-known fact that the dimension algebraic of convex sets on the unit sphere $S(X)$ of a normed linear space X controls the dimension of the set $P_M(x)$ of all best approximants from a linear subspace M to $x \in X$ [4, p. 38]. Such dimension of convex subsets of $S(X)$ is known among other things for $C(T)$, where T denotes a compact space, and for $L^p(\mu)$ spaces over a positive measure space (Ω, Σ, μ), where $1 \leq p \leq \infty$. Namely, $L^p(\mu)$ is strictly convex for $1 < p < \infty$, i. e. the only non-empty convex subsets in $S(L^p(\mu))$ are one-point sets. On the other hand, $C(T)$ as well as $L^p(\mu)$ for $p=1, \infty$ are already flat; more precisely $L^\infty(\mu)$ is flat iff it is not finite dimensional space, and $L^1(\mu)$ is flat iff (Ω, Σ, μ) is not purely atomic (see [2]). Recall that a Banach space X is thermed flat if there exists a rectifiable curve on $S(X)$ with antipodal points and of length two. This yields in the flat case of X that $S(X)$ contains a closed flat area (not weakly compact), which is the largest possible in the sense of diameter (R. Harrel, L. Karlovitz, 1970; cf. [2] for bibliography). For our purposes the main observation is that such flat areas cannot be finite dimensional.

In the best approximation theory the notion of k-strict convexity of a normed space has been introduced in connection with a dimension of the sets $P_M(x)$ by I. Singer in 1960; cf. [4]. The k-strict convexity of a normed space X means that $S(X)$ contains at most $k-1$ dimensional convex sets or equivalently for any $x_0, \ldots, x_k \in S(X) \| \Sigma_0^k x_i \| = k+1$ implies the linear dependence of these elements, where $1 \leq k < \infty$. Extending a little this notion we shall say X is ∞-strictly convex, if all convex subsets on $S(X)$ are finite dimensional.

Our purpose in this paper is to solve a problem of k-strict convexity (in the extended sense) of Fenchel-Orlicz spaces $L^{\Phi}(\mu, X)$ (see [6] for the theory of such spaces) over not purely atomic measure spaces (Ω, Σ, μ), endowed with the norm $\|f\|_\Phi = \inf\{r>0: \int_\Omega \Phi(f/r)d\mu \leq 1\}$, where $\Phi: X \to [0, \infty]$ is a Young's function and X denotes a normed space. Recall that $f \in L^\Phi(\mu, X)$, if $I_\Phi(\alpha f) = \int_\Omega \Phi(\alpha f)d\mu < \infty$ for some $\alpha > 0$ and $f: \Omega \to X$ is strongly measurable function. We say Φ is a Young's function, if it is even, convex function with $\Phi(0)=0$ and $\Phi(tx) \to \infty$, if $t \to \infty$ for some $x \in X$. Under the above definitions $L^\Phi(\mu, X)$ denotes a normed linear space. Let us mention that if $X_\Phi = \{x \in X: \Phi(\alpha x) < \infty$ for some $\alpha > 0\}$, then $f \in L^\Phi(\mu, X)$ implies that the image of f belongs to X_Φ almost everywhere. We shall assume that X_Φ contain at least two points.

Roughly speaking, we have found that for $L^\Phi(\mu, X)$ spaces with not purely atomic measure the situation does not differ substantially from that of $L^p(\mu)$ case, i. e. convex subsets on $S(L^\Phi(\mu, X))$ can be 1- or ∞-dimensional only. Let us notice that the background for such studies was given recently by Turret [5, 6, 3].

2. Results. First of all recall the result of B. Turret stated in [6].

Theorem 1. *Let* (Ω, Σ, μ) *be a measure space that is not purely atomic and let* X *be a normed space. Then the Fenchel-Orlicz space* $L^\Phi(\mu, X)$ *is strictly convex iff*

(i) $\int_\Omega \Phi(f/\|f\|_\Phi)d\mu = 1$ *for* $0 \neq f \in L^\Phi(\mu, X)$;

(ii) $\Phi|_{X_\Phi}$ *is strictly convex.*

A proof for the scalar case $X = \boldsymbol{R}$ is given in [5]. For a normed space does not differ substantially (cf. [1]).

Our main result here is the following theorem.

Theorem 2. *Let* X *be a normed linear space and let* (Ω, Σ, μ) *be a measure space that is not purely atomic. The Fenchel-Orlicz space* $L^\Phi(\mu, X)$ *is* ∞*-strictly convex iff the both conditions* (i), (ii) *from Theorem* 1 *hold.*

To give a proof we shall need the lemma, for which a basis states the Lemma 1 from [5]. For the sake of simplicity the proofs presented here concern the case of $X = \boldsymbol{R}$ only. We shall write in this case simply $L^\Phi(\mu)$. For the general case of normed space for X see [1].

Lemma 1. *If* (Ω, Σ, μ) *is not purely atomic,* Φ *is finite on* X_Φ *and* $I_\Phi(g) < 1$ *for some norm-one function* $g \in L^\Phi(\mu, X)$, *then there exists a sequence* $f_0, f_1, \ldots$ *with* $\|f_i\|_\Phi = 1$, $i = 0, 1, \ldots$, *that is linearly independent and such that for every nonnegative integer* $i_0, \ldots, i_k$ *and every* $\lambda_j \geq 0$, $\Sigma_0^k \lambda_j = 1$ *we have* $\|\Sigma_0^k \lambda_j f_{i_j}\|_\Phi = 1$.

Proof. The scalar case. First we show, as in [5, p. 466], that $\int_\Omega \Phi(g/r)d\mu = \infty$ for $r \in (0, 1)$. Let $A = \{t \in \Omega: g(t) \neq 0\}$. We distinguish three cases: $A \supset Q$, where $Q \in \Sigma$, $0 < \mu(Q) < \infty$ and Q contains no atoms; A is an infinite sum of atoms except a measure zero set; A is a finite sum of atoms except a measure zero set.

Let us consider the third case, when $A = \cup_0^s A_i$ (except a measure zero set), where A_i denotes atoms. We have $\mu(A) = \infty$. Otherwise $\int_\Omega \Phi(g)d\mu = \Sigma_0^s \Phi(x_i)\mu(B_i)$ and is strictly less than one by the assumption, where $\mu(B_i \Delta A_i) = 0$, $B_i \in \Sigma$. Since $\|g\|_\Phi = 1$, then $\Sigma_0^s \Phi(x_i/r)\mu(B_i) < 1$ for $r \geq 1$ and > 1 for $r < 1$. But such a sum is continuous function of r on $(0, \infty)$ and therefore takes the value one for $r = 1$, a contradiction. Hence $\mu(A) = \infty$.

It is easy to see that $\Phi(tx)=0$, if $0\leq t<1$, and differs from zero, if $t>1$, for some $x\in R$. Let $Q\subset\Omega\setminus A$ be any non-atomic measurable set such that $0<\mu(Q)<\infty$. Let $Q=\cup_0^\infty Q_i$, $\mu(Q_i)>0$, $Q_i\cap Q_j=\emptyset$. Define functions $f_i=x\chi_A+t_iy\chi_{Q_i}$, $i=0,1,\ldots$, where y and t_i are chosen in such a way that $\Phi(y)\neq 0$ and $\Phi(t_iy)=1/(2\mu(Q_i))$. These functions are linearly independent. Moreover $\int_\Omega\Phi(f_i)d\mu=\Phi(x)\mu(A)+\Phi(t_iy)\mu(Q_i)=1/2<1$ and $\int_\Omega\Phi(f_i/r)d\mu=\Phi(x/r)\mu(A)+\cdots=\infty$ for $r\in(0,1)$. Also for any different nonnegative integers $i_0,\ldots,i_k$ and $\lambda_j\geq 0, \Sigma_0^k\lambda_j=1$, we have $\int_\Omega\Phi(\Sigma_0^k\lambda_jf_{i_j})d\mu\leq\Sigma_0^k\lambda_j\int_\Omega\Phi(f_{i_j})d\mu<1$ and $\int_\Omega\Phi(\Sigma_0^k\lambda_jf_{i_j}/r)d\mu=\Phi(x/r)\mu(A)+\cdots=\infty$ for $r\in(0,1)$. Hence we conclude that $\|f_i\|_\Phi=1$ and $\|\Sigma_0^k\lambda_jf_{i_j}\|_\Phi=1$.

Now let us consider the both remaining cases: a) A contains the set of type Q; b) A is an infinite sum of atoms. We construct a sequence $f_0, f_1,\ldots$ as follows: let $f_{-1}=g$, $S_0=\Omega$ and put $f_i=f_{i-1}\chi_{S_i}$, where on every step $i=0,1,\ldots$ the set S_i is divided in two parts of positive measures T and $S_i\setminus T$ in such a manner that $f_i\neq 0$ on each of these parts, and each of them contains a set of type Q just as in the case a), or each of them contains infinitely many atoms just as in the case b) holds. Denote by S_{i+1} such a part, for which $\int_\Omega\Phi(f_i\chi_{S_{i+1}}/r)d\mu=\infty$ for $r\in(0;1)$. This is always available on account of the decomposition $\int_\Omega\Phi(f_i/r)d\mu=\int_\Omega\Phi(f_i\chi_T/r)d\mu+\int_\Omega\Phi(f_i\chi_{S_i\setminus T}/r)d\mu$, where at the first step we make use of the fact mentioned at the first line of the proof. Summarizing, we get functions $f_i=g\chi_{S_i}$, $i=0,1,\ldots$, on account of $S_0\supset S_1\supset\cdots$. Such functions are linearly independent and belong to $L^\Phi(\mu)$. Moreover, $\int_\Omega\Phi(f_i)d\mu<1$. Thus $\|f_i\|_\Phi=1$. Also it is easy to see that for any different nonnegative integers $i_0,\ldots,i_k$ and real $\lambda_j\geq 0$, $\Sigma_0^k\lambda_j=1$ we have $\int_\Omega\Phi(\Sigma_0^k\lambda_jf_{i_j})d\mu<1$, while $\int_\Omega\Phi(\Sigma_0^k\lambda_jf_{i_j}/r)d\mu=\infty$ for $r\in(0,1)$. Hence again $\|\Sigma_0^k\lambda_jf_{i_j}\|_\Phi=1$ and the lemma is completely proved.

Proof of the Theorem 2 (The scalar case $X=R$). Let $L^\Phi(\mu)$ be ∞-strictly convex. If one assumes $\Phi(x)=\infty$ for some $x\in R$, then, extending a method from [5], pp. 464-465, we may produce infinitely many functions of the form $f_i=\Sigma_{j=i+1}^\infty t_j\chi_{S_j}$, $i=0,1,\ldots$, which are linearly independent with $\|f_i\|_\Phi=1$ for all $i=0,1,\ldots$ and such that any convex combination of these functions is of norm one. Thus we obtain a contradiction with ∞-strict convexity. Therefore, Φ is finite for all x and hence continuous (as a Young's function).

Next, suppose that the condition (i) does not hold, i. e. for some norm-one $g\in L^\Phi(\mu)$ $\int_\Omega\Phi(g)d\mu<1$. Then, applying the Lemma 1, we obtain a contradiction with ∞-strict convexity of $L^\Phi(\mu)$. Thus the necessity of condition (i) is proved.

Finally suppose that the condition (ii) does not hold, i. e. there exist $y<x$ such that $\Phi(\lambda x+(1-\lambda)y)=\lambda\Phi(x)+(1-\lambda)\Phi(y)$ for $0\leq\lambda\leq 1$. Let S, T denote two disjoint elements of Σ containing no atoms, with finite, positive and sufficiently small measures, if necessary. Let $S=\cup_0^\infty S_j$, where S_j are pairwise disjoint sets and such that $0<\mu(S_i)<\mu(S_j)$, $j<i$. Let us define functions $f_i=x\chi_{Q_i}+y_i\chi_{Q\setminus Q_i}+z\chi_T$, $i=0,1,\ldots$, where y_i are chosen in such a manner that for $\lambda_i=\mu(S_i)/\mu(S)$ we have $\lambda_ix+(1-\lambda_i)y_i=v$, where

$i=0, 1, \ldots$ Without loss of generality assume $y_0=y$. Let us mention that y_i approaches v, if $i\to\infty$ because λ_i approaches zero. Moreover, z is such that $\Phi(z)\neq 0$ and $\Phi(v)\mu(S)+\Phi(z)\mu(T)=1$. These functions belong to $L^{\Phi}(\mu)$. Now, in view of the linearity of Φ between y and z, it is easy to check that $\int_{\Omega}\Phi(f_i)d\mu=1$ and hence $\|f_i\|_{\Phi}=1$ for $i=0, 1, \ldots$ Further, for any nonnegative and different integers $i_0, \ldots, i_k$ and for all $\lambda_j\geq 0$, $\Sigma_0^k\lambda_j=1$:

$$\sum_{j=0}^{k}\lambda_j f_{i_j}(t)=\begin{cases} z & \text{, if } t\in T,\\ 0 & \text{, if } t\in\Omega\setminus S\setminus T,\\ \sum_0^k\lambda_j y_{i_j} & \text{, if } t\in S\setminus\bigcup_0^k S_{i_j},\\ \sum_{0=j\neq l}^{k}\lambda_j y_{i_j}+\lambda_l x, & \text{if } t\in S_{i_l}. \end{cases}$$

Since Φ is linear between y and x, it follows that

$$\Phi\left(\sum_0^k\lambda_j f_{i_j}(t)\right)=\sum_0^k\lambda_j\Phi(f_{i_j}(t))$$

almost everywhere on Ω. Hence $\int_{\Omega}\Phi(\Sigma_0^k\lambda_j f_{i_j})d\mu=1$, so that $\|\Sigma_0^k\lambda_j f_{i_j}\|_{\Phi}=1$ for all $i_0, \ldots, i_k$ and λ_j as above.

To prove that these functions are linearly independent let $\Sigma_0^k\lambda_j f_{i_j}=0$ for some nonnegative integers $i_0, \ldots, i_k$. This leads to a system of equations with the determinant

$$\det\begin{bmatrix} x & y_{i_1} & \ldots & y_{i_k}\\ y_{i_0} & x & \ldots & y_{i_k}\\ \ldots & \ldots & \ldots & \ldots\\ y_{i_0} & y_{i_1} & \ldots & x \end{bmatrix}=\prod_{j=0}^{k}(x-y_{i_j})\left(1+\sum_{j=0}^{k}\frac{y_{i_j}}{x-y_{i_j}}\right).$$

By our assumptions on x and y_i, $i=0, 1\ldots$, the above determinant is different from zero and this yields the desired linear independence of $f_{i_0}, \ldots, f_{i_k}$ and hence $f_0, f_1, \ldots$

Thus again we get the contradiction with ∞-strict convexity of $L^{\Phi}(\mu)$, so that Φ must be strictly convex.

The theorem just proved, together with Theorem 1, yields the following corollary.

Corollary. *Let X be a normed linear space and let (Ω, Σ, μ) be a measure space that is not purely atomic. The following statements are equivalent:*

1) *$L^{\Phi}(\mu, X)$ is rotund;*
2) *$L^{\Phi}(\mu, X)$ is k-strictly convex for any fixed $k\geq 1$;*
3) *$S(L^{\Phi}(\mu, X))$ contains no infinite dimensional convex subset* (i. e. *$L^{\Phi}(\mu, X)$ is ∞-strictly convex*).

This corollary can be put into another form, if one applies a well-known connection of k-strict convexity with the existence of $(k-1)$-semi-Chebyshev subspaces, where $1\leq k<\infty$ [4, p. 38].

Corollary 2. *Let X and (Ω, Σ, μ) be such as in Theorem 2. The following statements are equivalent:*

1) *All linear subspaces of* $L^{\Phi}(\mu, X)$ *are semi-Chebyshev subspaces.*

2) *All linear subspaces of* $L^{\Phi}(\mu, X)$ *are k-semi-Chebyshev subspaces, where k is any fixed natural number such that* $1 \leq k < \infty$.

3) *All linear subspaces of* $L^{\Phi}(\mu, X)$ *are* ∞*-semi-Chebyshev subspaces.*

Let Y denote any normed linear space. We recall that a linear subspace $M \subset Y$ is called k-semi-Chebyshev subspace, if for all $y \in Y$ $\dim P_M(y) < k$, where $1 \leq k < \infty$ [4].

In the above corollary we have extended this notion up to the case $k = \infty$ (i. e. ∞-semi-Chebyshev subspace). Let us notice that the proximal linear subspace $M \subset Y$, that is (in our terminology) also ∞-semi-Chebyshev subspace, is called EF-subspace (P. Morris) or pseudo-Chebyshev-subspace [4].

Concerning the equivalence of statement 3) from the above corol) lary and statement 3) from the Corollary 1, it appears to be a consequence of the equalities $P_M(y) = M \cap S(y, \rho(y, M)) = y + (M - y) \cap S(y, \rho(0, M - y))$ and that of $A \subset H_A \cap S(0, \rho(0, H_A)) = P_{M_A}(0) = P_{H_A + z}(z) - z$, $-z \in H_A$ for any convex subset A on the unit sphere $S(Y)$, where $H_A = \mathrm{Aff}(A)$ and $S(y, \rho(y, M))$ denote the sphere in Y at y with the radius $\rho(y, M) = \inf\{\|y - h\| : h \in M\}$. Thus $S(Y) = S(0, 1)$, hovewer it does not lead to misunderstanding.

Let us mention that for $X = \mathbf{R}$ and for a finite measure space that is not purely atomic, it is known on $L^{\Phi}(\mu)$ something more. Namely, $L^{\Phi}(\mu)$ is flat iff $\Phi(\cdot) = \alpha|\cdot|$ for some $\alpha > 0$ or the condition (i) from Theorem 1 does not hold [3]. For such measure spaces the mentioned condition (i) is simply the so-called Δ_2 condition for large values of x. If one collects these facts tegether with those of Theorems 1 and 2, then they may be expressed in the following clear dependence.

Corollary 3. *Let* (Ω, Σ, μ) *be a finite measure space that is not purely atomic. Then, writing* L^{Φ} *instead of* $L^{\Phi}(\mu)$:

	Δ_2 *condition for large x holds;*	Δ_2 *condition for large x doesn't hold;*
Φ *contains no flat part:*	$S(L^{\Phi})$ *contains no flat part;*	L^{Φ} *is flat;*
Φ *contains a flat part:*	$S(L^{\Phi})$ *contains an infinite dimensional flat part;*	L^{Φ} *is flat;*
$\Phi(\cdot) = \alpha|\cdot|$:	L^{Φ} *is flat;*	L^{Φ} *is flat.*

We say that Φ contains a flat part whenever Φ is linear on some interval or $\Phi(x) = \infty$ for some x. Also, we say that $S(L^{\Phi})$ contains a flat part, if it contains at least one dimensional convex set. A similar result one may formulate for the general case of normed linear space X, cf. [1].

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