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ON THE EVALUATION OF DOUBLE INTEGRALS*

M. Levin

Summary. A cubature formula, consisting of line integrals, which is optimal on a set of functions, satisfying given boundary conditions, is obtained. The line integrals of this formula may be evaluated by optimal quadrature formulas. The advantage of this formula over the optimal cubature formula with a rectangular lattice of knots is shown. This approach to optimal cubatures is stimulated by the idea of blending [1-2].

1. Notations and Definitions. Let $L_{1i}f(\cdot)=0$, $i=1,\ldots,r_1$, and $L_{2j}f(\cdot)=0$, $j=1,\ldots,r_2$, be linear homogeneous boundary conditions for the interval $[0,1]$ such that the problems

$$(1)\quad \begin{cases} y^{(r)}(x)=0, \\ L_{1i}y(\cdot)=0, \quad i=1,\ldots,r_1, \end{cases} \quad \text{and} \quad \begin{cases} y^{(r)}(x)=0 \\ L_{2j}y(\cdot)=0, \quad j=1,\ldots,r_2, \end{cases}$$

have a unique solution $y(x)\equiv 0$, and let $g_1(x,t)$ and $g_2(x,t)$ be the Green functions corresponding to problems (1) [3].

Let $D=[0,1]\times[0,1]$, $1<p<\infty$, $p^{-1}+q^{-1}=1$. We consider the following sets of functions:

$W^r_{g_k}L_p=\{f(x): f^{(r)}(x)$ piecewise continuous on $[0,1]$,

$$\|f^{(r)}(\cdot)\|_{L_p(0,1)}\leq 1, \quad L_{ki}f(\cdot)=0, \quad i=1,\ldots,r_k\}, \quad k=1,2,$$

$W^{2r}_{g_1g_2}L_p=\{f(x,y): f^{(l+s)}_{x^ly^s}(x,y)\,(l,s\leq r)$ piecewise continuous on D,

$$\|f^{(2r)}_{x^ry^r}(\cdot,\cdot)\|_{L_p(D)}\leq M,\ L_{1i}f(\cdot,y)\equiv 0,\ i=1,\ldots,r_1;\ L_{2j}f(x,\cdot)\equiv 0,\ j=1,\ldots,r_2\}.$$

The quadrature formula

$$(2)\qquad \int_0^1 f(x)dx=\sum_{k=1}^{m}A_kf(x_k)+r(f), \quad 0\leq x_1<\ldots<x_m\leq 1$$

is called optimal for the set H of functions $f(x)$, if the coefficients A_k and knots x_k of the formula are chosen so that the quantity

$$(3)\qquad \sup\{|r(f)|: f\in H\}$$

is minimal.

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Designate

$$A_{p1}^{(m)}, \dots, A_{pm}^{(m)}; \quad x_{p1}^{(m)}, \dots, x_{pm}^{(m)}; \quad r_{p1}^{(m)},$$

$$B_{p1}^{(m)}, \dots, B_{pm}^{(m)}; \quad y_{p1}^{(m)}, \dots, y_{pm}^{(m)}; \quad r_{p2}^{(m)},$$

the coefficients, knots and the value (3) of the optimal formulas (2) for the sets $W_{g_1}^r L_p$ and $W_{g_2}^r L_p$, respectively. It is known [3] that $r_{p_1}^{(m)} = 0(m^{-r})$, $r_{p2}^{(m)} = 0(m^{-r})$.

The formula

$$\int_0^1 \int_0^1 f(x, y)dxdy = \sum_{k=1}^{m} \sum_{l=1}^{m} C_{kl} f(x_k, y_l) + R(f), \tag{4}$$

$$0 \leq x_1 < \dots < x_m \leq 1, \quad 0 \leq y_1 < \dots < y_m \leq 1$$

is called the optimal formula for the set F of functions $f(x, y)$, if its coefficients and knots are chosen so that the quantity $R = \sup\{|R(f)| : f \in F\}$ has the least value.

It is shown in [3, 4] that the optimal formula (4) on the set $W_{g_1g_2}^{2r} L_2$ has the coefficients $C_{kl} = A_{2k}^{(m)} B_{2l}^{(m)}$, the knots $x_k = x_{2k}^{(m)}$, $y_l = y_{2l}^{(m)}$ and the remainder $R = 0(m^{-r})$.

In the case $p \neq 2$ the optimal formula (4) on the set $W_{g_1g_2}^{2r} L_p$ is not yet found.

2. The Optimal Cubature Formula. We will find an optimal formula of the form

$$\int_0^1 \int_0^1 f(x, y)dxdy = \sum_{k=1}^{n} \alpha_k \int_0^1 f(x_k, y)dy + \sum_{j=1}^{n} \beta_j \int_0^1 f(x, y_j)dx \tag{5}$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{n} \gamma_{kj} f(x_k, y_j) + E(f),$$

$$0 \leq x_1 < \dots < x_n \leq 1, \quad 0 \leq y_1 < \dots < y_n \leq 1,$$

for the set $W_{g_1g_2}^{2r} L_p$. In other words, we will find the formula (5) with the least value of $E = \sup\{|E(f)| : f \in W_{g_1g_2}^{2r} L_p\}$.

Theorem. *The coefficients and knots*

$$\alpha_k = A_{pk}^{(n)}, \quad \beta_j = B_{pj}^{(n)}, \quad \gamma_{kj} = -A_{pk}^{(n)} B_{pj}^{(n)},$$

(6)

$$x_k = x_{pk}^{(n)}, \quad y_j = y_{pj}^{(n)}, \quad k, j = 1, \dots, n,$$

and the estimate

$$E = M r_{p1}^{(n)} r_{p2}^{(n)} \tag{7}$$

are the coefficients, knots and estimate of the optimal formula (5) *on the set* $W_{g_1g_2}^{2r} L_p$.

Proof. Let $f(x, y) \in W^{2r}_{g_1 g_2} L_p$, then [3]

$$f(x, y) = \int_0^1 \int_0^1 f^{(2r)}_{x^r y^r}(t, u) g_1(x, t) g_2(y, u)\, dt\, du.$$

Hence by (5)

$$E(f) = \int_0^1 \int_0^1 f^{(2r)}_{x^r y^r}(t, u) K(t, u)\, dt\, du, \tag{8}$$

where

$$K(t, u) = \varphi_1(t)\varphi_2(u) - \varphi_2(u) \sum_{k=1}^{n} \alpha_k g_1(x_k, t) - \varphi_1(t) \sum_{j=1}^{n} \beta_j g_2(y_j, u)$$

$$- \sum_{k=1}^{n} \sum_{j=1}^{n} \gamma_{kj} g_1(x_k, t) g_2(y_j, u),$$

$$\varphi_l(v) = \int_0^1 g_l(x, v)\, dx, \quad l = 1, 2.$$

Using Hölder's inequality, we obtain from (8) that

$$|E(f)| \leq M \| K(\cdot, \cdot) \|_{L_q(D)}. \tag{9}$$

Since the function

$$f_0(x, y) = \frac{M}{\| K(\cdot, \cdot) \|^{q/p}_{L_q(D)}} \int_0^1 \int_0^1 | K(t, u) |^{q-1} \operatorname{sgn} K(t, u) g_1(x, t) g_2(y, u)\, dt\, du$$

belongs to the set $W^{2r}_{g_1 g_2} L_p$, it follows from (8) that $E(f_0) = M \| K(\cdot, \cdot) \|_{L_q(D)}$. Then from (9) we have the equality $E = M \| K(\cdot, \cdot) \|_{L_q(D)}$.

By the result on 'polynomials' of least deviation from zero [5] we have that

$$\inf_{\{\alpha_k, \beta_j, \gamma_{kj}, x_k, y_j\}} \| K(\cdot, \cdot) \|_{L_q(D)} \tag{10}$$

$$= \inf_{\{\alpha_k, x_k\}} \| \varphi_1(\cdot) - \sum_{k=1}^{n} \alpha_k g_1(x_k, \cdot) \|_{L_q(0,1)} \cdot \inf_{\{\beta_j, y_j\}} \| \varphi_2(\cdot) - \sum_{j=1}^{n} \beta_j g_2(y_j, \cdot) \|_{L_q(0,1)}.$$

As the equalities

$$\inf_{\{\alpha_k, x_k\}} \| \varphi_1(\cdot) - \sum_{k=1}^{n} \alpha_k g_1(x_k, \cdot) \|_{L_q(0,1)} = r^{(n)}_{p1}$$

$$\inf_{\{\beta_j, y_j\}} \| \varphi_2(\cdot) - \sum_{j=1}^{n} \beta_j g_2(y_j, \cdot) \|_{L_q(0,1)} = r^{(n)}_{p2}$$

hold and are achieved by the coefficients and knots (6), it follows from (10) that the numbers (6) and (7) are the coefficients, knots and the estimate of the optimal formula (5) for the set $W^{2r}_{g_1 g_2} L_p$.

The theorem is proved.

Now let $\sup_{0\le y\le 1}\|f^{(r)}_{x^r}(\cdot, y)\|_{L_p(0,1)}\le M_1$, $\sup_{0\le x\le 1}\|f^{(r)}_{y^r}(x, \cdot)\|_{L_p(0,1)}\le M_2$. Then $M_1^{-1} f(x, y^{(n)}_{pj})\in W^r_{g_1}L_p$, $M_2^{-1} f(x^{(n)}_{pk}, y)\in W^r_{g_2}L_p$.

Applying optimal quadrature formulas for these sets to the integrals in (5) we obtain

$$(11)\qquad \int_0^1 f(x, y^{(n)}_{pj})dx=\sum_{k=1}^{n^2} A^{(n^2)}_{pk} f(x^{(n^2)}_{pk}, y^{(n)}_{pj})+r_1(f),\quad |r_1(f)|\le M_1 r^{(n^2)}_{p1},$$

$$(12)\qquad \int_0^1 f(x^{(n)}_{pk}, y)dy=\sum_{j=1}^{n^2} B^{(n^2)}_{pj} f(x^{(n)}_{pk}, y^{(n^2)}_{pj})+r_2(f),\quad |r_2(f)|\le M_2 r^{(n^2)}_{p2}.$$

Designating $a_n=|A^{(n)}_{p1}|+\ldots+|A^{(n)}_{pn}|$, $b_n=|B^{(n)}_{p1}|+\ldots+|B^{(n)}_{pn}|$ and substituting (11) and (12) into the optimal formula (5) for the set $W^{2r}_{g_1g_2}L_p$, we obtain the following cubature formula:

$$(13)\qquad \int_0^1\int_0^1 f(x, y)dxdy=\sum_{k=1}^{n}\sum_{j=1}^{n^2} A^{(n)}_{pk} B^{(n^2)}_{pj} f(x^{(n)}_{pk}, y^{(n^2)}_{pj})$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{n^2} B^{(n)}_{pj} A^{(n^2)}_{pk} f(x^{(n^2)}_{pk}, y^{(n)}_{pj})-\sum_{k=1}^{n}\sum_{j=1}^{n} A^{(n)}_{pk} B^{(n)}_{pj} f(x^{(n)}_{pk}, y^{(n)}_{pj})+E_1(f),$$

where

$$(14)\qquad |E_1(f)|\le a_n M_2 r^{(n^2)}_{p2}+b_n M_1 r^{(n^2)}_{p1}+M r^{(n)}_{p1} r^{(n)}_{p2}.$$

It follows from the convergence of formulas (2) that a_n and b_n are bounded as $n\to\infty$. Hence it follows from (14) that $|E_1(f)|=O(n^{-2r})$. The formula (13) has a remarkable advantage over the optimal formula (4). The optimal formula (4) with $m=n^2$ has the error estimate $O(n^{-2r})$ and it uses n^4 point values of $f(x, y)$, while the formula (13) has the same error estimate $O(n^{-2r})$, but uses only $2n^3+n^2$ point values of the function $f(x, y)$.

3. Example. We compare the evaluation of the integral

$$I=\int_0^1\int_0^1 \frac{(x-x^2)(y-y^2)}{0.2+xy}dxdy=0.0701598\ldots$$

by the optimal formula (4) with $m=n^2$ and by the formula (13) obtained from the optimal formula (5). Both formulas are taken for the set $W^{2r}_{g_1g_2}L_2$.

As the integrand $f(x, y)$ satisfies the conditions $f(0, y)=f(1, y)=f(x, 0)=f(x, 1)=0$, we can take the functions $g_1(x, t)=g_2(x, t)$ as Green's functions for the problem $y''=0$, $y(0)=y(1)=0$. Then by [3] we have

$$A^{(m)}_{2k}=B^{(m)}_{2k}=2\varepsilon,\quad k=2,\ldots, m-1,$$

$$A^{(m)}_{21}=B^{(m)}_{21}=A^{(m)}_{2m}=B^{(m)}_{2m}=(1+1.25\sqrt{2/3})\varepsilon,$$

$$x^{(m)}_{2k}=y^{(m)}_{2k}=2\varepsilon(\sqrt{2/3}+k-1),\quad k=1,\ldots, m,$$

$$\varepsilon=0.5(2\sqrt{2/3}+m-1)^{-1}.$$

Let I_n be an approximate value of I, obtained by the optimal formula (4) with the $k_n = n^4$ knots and let I'_n be the approximate value obtained by formula (13) with $q_n = 2n^3 + n^2$ knots.

We obtain for $n = 4, 7, 9$

I'_n	I_n	q_n	k_n	n
0.0701302	0.0701319	144	256	4
0.0701587	0.0701588	735	2401	7
0.0701596	0.0701596	1539	6561	9

The superiority of the formula (13) over the optimal formula (4) is obvious.

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Tel-Aviv University
Department of Mathematical Sciences
Ramat-Aviv, Tel Aviv Israel

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