КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'81
София, 1983, с. 419—424

CONSTRUCTIVE FUNCTION THEORY'81
Sofia, 1983, p. 419—424

ON THE MOMENT PROBLEM AND THE CONVERGENCE OF PADE APPROXIMANTS FOR MEROMORPHIC FUNCTIONS OF STIELTJES TYPE

G. López

Summary. Let $f=\widehat{\mu}+r$, where μ is a positive measure with support in $[0,+\infty[$, $\widehat{\mu}=\mu * z^{-1}$ and r a rational function with real coefficients, $r(\infty)=0$. In case $r\equiv 0$ a well-known result of Stieltjes asserts the convergence to $\widehat{\mu}(\equiv f)$ of the diagonal Padé approximants associated to this function under the condition that the moment problem for $\{c_\nu\}$, $\nu\in N$, $c_\nu=\int t^\nu d\mu(t)$ is determined. In this paper we show that a condition, similar to that of Stieltjes in terms of certain moment problems, also assures the convergence of Padé approximants to f for general r. Our result reduces to that of Stieltjes, when $r\equiv 0$.

1. Introduction.

Let

$$f(z)=\sum_{\nu=0}^{\infty} f_\nu/z^{\nu+1} \tag{1}$$

be a formal power series. For each n, $m\in N$ there exist polynomials p, q of degree at most $n+m$ and n, respectively, $q\not\equiv 0$, such that

$$z^m q(z) f(z)-p(z)=A_{n,m}z^{-n-1}+\cdots, \tag{2}$$

where on the right-hand side we have a formal power series on increasing powers of z^{-1}. The quotient $p/z^m q$ of any such polynomials is determined uniquely and is called the $(n+m,\ n)$ Padé approximant associated to f and in the sequel is denoted by $\pi_{n+m,n}$. For more details see [1] and [2].

Let μ be a positive measure, whose support supp μ, contained in $[0,+\infty[$, consists of infinitely many points* and whose moments $c_\nu=\int x^\nu d\mu(x)$, $\nu=0, 1, 2, \dots$, are all finite. Let $\widehat{\mu}(z)=\int(z-x)^{-1}d\mu(x)$. Obviously, $\widehat{\mu}$ is holomorphic in $D=C\setminus[0,\ +\infty[$. In the following we will consider functions $f=\widehat{\mu}+r$, where r is an arbitrary rational function with real coefficients and poles in $D(r(\infty)=0)$. For each such function we can take its asymptotic expansion of type (1) at $z=\infty$ (for instance, as $z\to\infty$, $z<0$) and construct for n, $m\in N$ the corresponding Padé approximants.

* If supp μ has only a finite number of points, then f is a rational function and the main result of this paper is trivial.

In [3] I proved that if

$$\sum_{\nu \geqq 1} c_\nu^{-1/2\nu} = \infty, \tag{3}$$

then, for $m=0$ fixed, the sequence $\{\pi_{n,n}\}$, $n \in N$, converges to f uniformly on every compact set $K \subset D' = D \setminus [r(z)=\infty]$. This result contains classical ones of Markov [4] (see also [5]) and Carleman [6] and a recent one of Rajmanov [7] (for more details see § 1 of [3]). Stieltjes proved in [8] that a sufficient condition for the convergence of $\pi_{n,n}$ to $\widehat{\mu}(r=0)$ is that the moment problem for the sequence $\{c_\nu\}$, $\nu \in N$, is determined, while Carleman showed in [6] that condition (3) yields the determinacy of such a moment problem. From Carleman's result it is obvious that whenever l is a polynomial, positive on $[0, +\infty[$, then (3) implies the determinacy of the moment problem for the sequence $c_\nu^* = \int x^\nu l(x) d\mu(x)$, $\nu \in N$, since (3) is equivalent to $\Sigma_{\nu \geqq 1} (c_\nu^*)^{-1/2\nu} = \infty$. The object of this paper is to prove, for each $m \in N$ fixed, a similar result to that, proved before by us, but under weaker conditions in terms of moment problems, which reduces to Stieltjes theorem in case $r=0$ and $m=0$.

Theorem. *Let μ be a positive measure such that the convex hull of* supp μ *is* $[0, +\infty[$; $f=\widehat{\mu}+r$, *where* $r=s_d/t_d$ *is an arbitrary rational function with real coefficients and poles in* D ($\deg t_d = d$, $r(\infty)=0$). *If for each polynomial l positive on* $[0, +\infty[$ *such that* $\deg l \leqq d$ *and* $m \in N$ *fixed the moment problem for the sequence* $c_\nu^* = \int x^\nu x^m (t_d l)(x) d\mu(x)$, $\nu \in N$ *is determined, then the sequence* $\{\pi_{n+m,n}\}$, $n \in N$, *satisfies:*

i) *For all n sufficiently large the number of poles of $\pi_{n+m,n}$ in D is equal to the number of poles of r; the poles of r in D tend as $n \to \infty$ to the poles of r in such a way that each pole of r pulls towards itself as many poles of $\pi_{n+m,n}$ as its multiplicity.*

ii) *The sequence* $\{\pi_{n+m,n}\}$, $n \in N$, *converges uniformly to f on every compact* $K \subset D' = D \setminus [r(z)=\infty]$.

We wish to point out that the proof of a similar result in case r has complex coefficients remains open even for nice measures. A very interesting paper in this direction is due to Gončar [9], where he considers measures, whose support consists of a closed segment $[a, b]$, and fulfils other additional properties true in particular, if $\mu' > 0$ with respect to Lebesgue measure in $[a, b]$. In [7] Rajmanov showed that for very simple measures, whose support consists of two segments, the sequence $\{\pi_{n,n}\}$, $n \in N$ does not necessarily converge in D'. In Section 2 some auxiliary lemmas are stated. Proofs are omitted, since they are simple and resemble those of similar lemmas, proved in [3]. The demonstration of the main result is in Section 3. In the following we will keep the notations introduced above.

2. Lemmas. 1. For each $n > d$ and $m \in N$ there exist polynomials p_{n+m}, q_n such that

$$\deg p_{n+m} \leqq n+m-1, \ \deg q_n \leqq n, \quad q_n \not\equiv 0; \tag{4}$$

$$\int x^\nu x^m (q_n t_d)(x) d\mu(x) = 0, \quad \nu = 0, 1, \ldots, n-d-1; \tag{5}$$

$$\int \frac{z^m (q_n t_d)(z) - x^m (q_n t_d)(x)}{z-x} d\mu(x) + z^m (q_n s_d)(z) - (t_d p_{n+m})(z) = 0. \tag{6}$$

The existence of such polynomials reduces to solving a homogeneous system of $2n+m$ linear equations with $2n+m+1$ unknowns, for which a non-trivial solution always exists. This fact guarantees that q_n can be chosen such that $q_n \neq 0$. In the following we will suppose that q_n is monic, that is that the coefficient of its term of highest power is 1.

Lemma 1. *Let $n>d$ and p_{n+m}, q_n be arbitrary polynomials, satisfying the system* (4)—(6). *Then $\pi_{n+m,n}=p_{n+m}/z^m q_n$ and*

$$(7)\qquad (f-\pi_{n+m,n})(z)=\int \frac{x^m(hq_nt_d)(x)}{z^m(hq_nt_d)(z)}\,\frac{d\mu(x)}{z-x},$$

where h is an arbitrary polynomial such that $\deg h \leq n-d$.

Formula (7), writing $p_{n+m}/z^m q_n$ instead of $\pi_{n+m,n}$, is a direct consequence of (4)—(6). On the other hand, from that formula it is obvious that $\pi_{n+m} = p_{n+m}/z^m q_n$ from the definition of $\pi_{n+m,n}$.

2. Let $n>d$ and p_{n+m}, q_n as above. From (5) we get that q_n has in $]0, +\infty[$ at least $n-d$ changes of sign (compare with [5, p. 57]). Consequently q_n can be written as $q_n=q_{n,1}\cdot q_{n,2}$, where $\deg q_{n,1}=n-d_n=n'$, $d_n\leq d$, moreover the zeros $\{x_{n,i}\}$, $i=1, 2, \ldots, n-d_n$ of $q_{n,1}$ are all simple and lie on $]0, +\infty[$, while $q_{n,2}$ does not change sign on $[0, +\infty[$, $\deg q_{n,2}\leq d_n \leq d$. We will denote

$$q_{n,1}(z)=\prod_{i=1}^{n'}(z-x_{n,i})\,;\quad d\mu_n = x^m q_{n,2} t_d\, d\mu$$

and

$$p_{n,1}(z)=\int (z-x)^{-1}(q_{n,1}(z)-q_{n,1}(x))\,d\mu_n(x).$$

Without loss of generality we can suppose that $d\mu_n \geq 0$.

Lemma 2. *For each $n>d$ the following formulas hold:*

$$(8)\qquad z^m(q_{n,2}t_d)(f-\pi_{n+m,n})(z)=\int \frac{d\mu_n(x)}{z-x}-p_{n,1}q_{n,1}^{-1}(z),\quad z\in D;$$

$$(9)\qquad \frac{p_{n,1}}{q_{n,1}}(z)=\sum_{i=1}^{n'}\frac{\lambda_{n,i}}{z-x_{n,i}},\quad \lambda_{n,i}=\int \frac{q_{n,1}(x)\,d\mu_n(x)}{q'_{n,1}(x_{n,i})(x-x_{n,i})}.$$

Formula (8) is a convenient rearrangement of (7), taking in regard the above defined polynomials $p_{n,1}$ and $q_{n,1}$. Relations (9) are easily obtained by direct calculations of the residues of $p_{n,1}/q_{n,1}$, having in mind the integral formula for $p_{n,1}$.

3. Let us consider the linear functional Λ_n, $n>d$, defined on the set of continuous functions $\subset[0, +\infty[$ by the formula $\Lambda_n(\varphi)=\sum_{i=1}^{n'}\lambda_{n,i}\varphi(x_{n,i})$, $\varphi\in\subset[0, +\infty[$. The following lemma is an analogue of the Gauss-Jacobi quadrature formula (see [5, p. 60]).

Lemma 3. *For each $n>d$ and every polynomial P, $\deg P<2n-(d+d_n)$ we have that*

$$(10)\qquad \int P(x)\,d\mu_n(x)=\Lambda_n(P).$$

From Lemma 3 we have

Lemma 4. *For each $n>d$ the number of positive $\lambda_{n,i}$ is not less than $n-(d+d_n)/2$.*

This follows inmediately, taking in (10) a convenient P.

3. Proof of the Theorem. 1. Let $\rho_n(z)=\Pi^-(z-x_{n,i})^2$ where Π^- stands for the product over those i, for which $\lambda_{n,i}<0$. From Lemma 4 is deduced that $\deg \rho_n \leq 2(n'-n+(d+d_n)/2)=d-d_n$.

Let $R=2\max\{|z|: t_d(z)=0\}$, $M=\max\{2R, 1\}$ and B_n be the product of all the zeros of $q_{n,2}$ and f_n, whose module is greater than M (each zero taken according to its multiplicity). Everywhere in the sequel $n>d$. We schall now prove that the sequence of functions $\Phi_n=B_n^{-1}z^m\rho_n t_d q_{n,2}(f-\pi_{n+m,n})$, $n\in N$, is uniformly bounded on each compact $K\subset D$.

From (8), (9) and the definition of Λ_n we obtain

$$\Phi_n(z)=B_n^{-1}\rho_n(z)\left[\int (z-x)^{-1}d\mu_n(x)-\Lambda_n((z-x)^{-1})\right]. \tag{11}$$

On the other hand

$$\frac{1}{z-x}-\frac{\rho_n(x)}{\rho_n(z)}\frac{1}{z-x}=\frac{\rho_n(z)-\rho_n(x)}{(z-x)\rho_n(z)}=K_n(x;\ z),$$

where $K_n(x, z)$ is a polynomial of x such that $\deg K_n<d-d_n<2n-(d+d_n)$. From (10) and (11) it follows that

$$\Phi_n(z)=B_n^{-1}\rho_n(z)=\left[\int((z-x)^{-1}-K_n(x;\ z))d\mu_n(x)\right. \tag{12}$$

$$\left.-\Lambda_n((z-x)^{-1}-K_n(x;\ z))\right]=B_n^{-1}\left[\int(z-x)^{-1}\rho_n(x)d\mu_n(x)-\Lambda_n((z-x)^{-1}\rho_n(x))\right],$$

hence for $z\in K\subset D$, K compact, from (12) and (10) we get

$$|\Phi_n(z)|\leq|B_n|^{-1}\int|z-x|^{-1}\rho_n(x)d\mu_n(x)+|B_n|^{-1}\Lambda_n(|z-x|^{-1}\rho_n(x))$$

$$\leq 2|B_n|^{-1}d^{-1}(K, [0, +\infty[)\int\rho_n(x)d\mu_n(x)\leq 2Cd^{-1}(K, [0, +\infty[),$$

where $d(K, [0, +\infty[)=\inf\{|z-x|: z\in K, x\in[0, +\infty[\}$, $C=\int(x+M)^{2d+m}d\mu(x)$

2. We shall now prove that $\{\Phi_n\}$, $n\in N$, converges uniformly to zero on every compact $K\subset D$. Since $\{\Phi_n\}$, $n\in N$, is uniformly bounded, it suffices to show that each convergent subsequence converges pointwise to zero in D. Let $\{\Phi_n\}$, $n\in L\subset N$ be a convergent subsequence. From this subsequence we can chose a subsequence $\{\Phi_n\}$, $n\in L'\subset L$ such that $B_n^{-1}\rho_n q_{n,2}$ tends as $n\to\infty$, $n\in L'$, to a polynomial l, $\deg l\leq d$, since its coefficients are uniformly bounded.

From (10) we see that for each fixed $\nu\in N$ and $n>\nu+d/2$, $n\in L'$

$$\sum_{i=1}^{n'} B_n^{-1}\lambda_{n,i}\rho_n(x_{n,i})x_{n,i}^{\nu}=\int x^{\nu}(B_n^{-1}x^m\rho_n q_{n,2}t_d)(x)d\mu(x) \tag{13}$$

$$\xrightarrow[n\to\infty]{}\int x^{\nu}x^m(lt_d)(x)d\mu(x)=c_{\nu}^{*}.$$

Since the moment problem for $\{c_{\nu}^{*}\}$, $\nu\in N$, is determined, then (13) yields (see [10, p. 119-121]) that for each continuous on $[0, +\infty[$ function φ and $n\in L'$

$$\Lambda_n(B_n^{-1}\rho_n\varphi)\xrightarrow[n\to\infty]{}\int\varphi(x)x^m(lt_d)(x)d\mu(x)\ \text{and}$$

$$\int\varphi(x)B_n^{-1}\rho_n(x)d\mu_n(x)\xrightarrow[n\to\infty]{}\int\varphi(x)x^m(lt_d)(x)d\mu(x).$$

In view of (12), taking $\varphi(x)=(z-x)^{-1}$, $z\in D$, this implies that $\Phi_n(z)\xrightarrow[n\to\infty]{}0$, which is what we wanted to prove.

3. Let K be a fixed compact set $K\subset D_R=D\cap[|z|<R]$ and $q_{n,R}(z)=\Pi(z-\xi_{n,i})$, where the product is taken over those zeros of $q_{n,2}$ inside D_R. From the definition of Φ_n it follows that $(f-\pi_{n+m,n})(z)=B_n\Phi_n(z)(z^m\rho_n q_{n,2}t_d)^{-1}(z)$. From this relation the following estimate readily follows:

$$|(f-\pi_{n+m,n})(z)|\leq A(K)\|\Phi_n\|_K|(q_{n,R}t_d)(z)|^{-1},\quad z\in K, \tag{14}$$

where $A(K)>0$ doesn't depend on n, and $\|\cdot\|_K$ is the sup-norm. From (14) we have that

$$\{z\in K:\ |(f-\pi_{n+m,n})(z)|\geq\varepsilon\}\subset\{z\in K:|(q_{n,R}t_d)(z)|\leq\varepsilon^{-1}A(K)\|\Phi_n\|_K\}.$$

Since $\|\Phi_n\|_K\xrightarrow[n\to\infty]{}0$ and $\deg q_{n,R}t_d\leq 2d$, then for all n sufficiently large (see 11, p. 291])

$$\operatorname{cap}\{z\in K:\ |(f-\pi_{n+m,n})(z)|\geq\varepsilon\}\leq(\varepsilon^{-1}A(K)\|\Phi_n\|_K)^{1/2d},$$

where cap(·) denotes the logarithmic capacity. Since the right-hand side tends to zero as $n\to\infty$, the sequence $\{\pi_{n+m,n}\}$, $n\in N$, converges in capacity to f on every compact $K\subset D_R$. The poles of $\pi_{n+m,n}$ in D_R are less than d for each $n\in N$ and the poles of f in D_R are exactly d. From Lemma 1 of [12] follows that the number of poles of $\pi_{n+m,n}$ in D is d for all sufficiently large n (in the following we consider only such n), moreover each pole of f in D pulls towards it as $n\to\infty$ as many poles of $\pi_{n+m,n}$ as its multiplicity. Hence $q_{n,R}=q_{n,2}\to t_d$ as $n\to\infty$. We can now assert that $\deg q_{n,2}=d$, $\deg q_{n,1}=n'=n-d$ and consequently from Lemma 4 we see that for all large $n\in N$, $\lambda_{n,i}>0$, $B_n=1$ and $\rho_n=1$. Part i) of the theorem has been proved. This implies that for all large n

$$(f-\pi_{n+m,n})(z)=\Phi_n(z)(q_{n,2}t_d)^{-1}(z),\quad z\in D. \tag{15}$$

Since for each $K\subset D'$, $\|\Phi_n\|_K\xrightarrow[n\to\infty]{}0$ and $q_{n,2}\xrightarrow[n\to\infty]{}t_d$ ii) easily follows from (15).

Inmediate consequences of the proved theorem are

Corollary 1. *If* $\Sigma_{n\geq1}c_n^{-1/2n}=\infty$, *then for each fixed* $m\in N$ $\pi_{n+m,n}$ *converges uniformly to* f *on every compact set* $K\subset D'$ *as* $n\to\infty$.

This result appears in [13] for the case, when $r=0$ (that is $f\equiv\hat{\mu}$).

Corollary 2. *If* $f\equiv\hat{\mu}$ *and the moment problem for* $\{c_\nu\}$, $\nu\geq k$, *is determined, then for each fixed* $m=0, 1,\dots,k$ *the sequences* $\{\pi_{n+m,n}\}$, $n\in N$, *converge to* $\hat{\mu}$ *uniformly on every compact* $K\subset D'$ *as* $n\to\infty$.

REFERENCES

1. O. Perron. Die Lehre von den Kettenbruchen. II. Stuttgart, 1957.
2. H. S. Wall. Analytic Theory of Continued Fractions. New York, 1948.
3. Г. Лопес. О сходимости аппроксимаций Паде для мероморфных функций стильтьесовского типа. *Мат. сборник*, **3**, 1980, 308-316.
4. А. А. Марков. Избранные труды по теории непрерывных дробей и теории функций, наименее уклоняющихся от нуля. Москва, 1948.
5. Г. Сегё. Ортогональные многочлены. Москва, 1948.
6. I. Carleman. Les fonctions quasi-analytic. Paris, 1926.

7. Е. А. Рахманов. О сходимости диагональных аппроксимаций Паде. *Мат. сборник*, **104**, 1977, 271-291.
8. Т. Стильтьес. Исследования о непрерывных дробях. Харьков, 1936.
9. А. А. Гончар. О сходимости аппроксимаций Паде для некоторых классов мероморфных функций. *Мат. сборник*, **97**, 1975, 608-629.
10. C. Radhakrishna Rao. Linear Statistical Inference and Its Applications. 2nd ed. New York, 1973.
11. Г. М. Голузин. Геометрическая теория функций комплексного переменного. Москва, 1966.
12. А. А. Гончар. О сходимости обобщенных аппроксимаций Паде мероморфных функций. *Мат. сборник*, **98**, 1975, 564-577.
13. J. Karlson. B. von Sydow. The convergence of Padé approximants to series of Stieltjes. *Ark. Mat.*, **14**, 1976, 43-53.

Havana University
Department of Mathematics
Havana Cuba

Received on June 4, 1981