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PROBABILISTIC REGULARITY OF BIRKHOFF MATRICES WITH ALL ACTIVE KNOTS

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Summary. The probability that the incidence matrix of a Birkhoff interpolation be regular, if it is chosen at random, is considered. In contrast to the problem, treated by G. G. Lorentz, R. A. Lorentz and S. D. Riemenschneider, it is assumed that each knot of interpolation must appear at least once in the incidence matrix, i. e. each knot is active.
It is shown that also with this additional assumption the probability that an incidence matrix, chosen from the class of all $m\times n$ incidence matrices be regular, becomes vanishingly small as m and n become infinitely large. If one makes the additional assumption that $m>\lambda n$ for some $\lambda>1/2$, the same holds true for the class of all incidence matrices, satisfying the Birkhoff condition.

The Birkhoff interpolation problem for polynomials of degree $n-1$ is the problem of finding, for a given $m\times n$ matrix $E=(e_{ij})$, $i=1,\dots,m$; $j=0, 1,\dots, n-1$, whose entries e_{ij} take the values 0 or 1, for given data b_{ij} and for given knots $a\leqq x_1<\cdots<x_m\leqq b$, that polynomial P, which satisfies

$$P^{(j)}(x_i)=b_{ij}, \tag{1}$$

for those pairs (i, j), for which $e_{ij}=1$. This is thus a generalization of Lagrange and Hermite interpolation. We use the terminology of [4], which should also be referred to for standard results on Birkhoff interpolation.

The matrix E is called the incidence matrix associated with the above interpolation problem. If, for a given matrix E, the problem possesses a unique solution for each choice of knots and for each choice of data, E is said to be regular. Otherwise it is singular.

There are two approaches for determining whether an incidence matrix is regular. One consists in analyzing the determinant to be inverted in solving this linear system. The other method, initiated by Schoenberg, is to come to a conclusion only on the basis of the structure of the incidence matrix. This method has yielded many useful partial solutions, but not any complete one. Up to now no necessary and sufficient conditions for the regularity of an incidence matrix are known, which are based only on its structure.

In view of this, Lorentz and Riemenschneider [3] proposed investigating the question of regularity of entire classes of incidence matrices. They posed the question: What proportion of the whole class of

$m\times n$ incidence matrices is regular? Another way of posing this question is to ask: If I choose an $m\times n$ incidence matrix at random, with what probability will it be regular? This question and its ramifications will be treated in this paper under the supplementary condition that each knot actually appears in the linear system 1. This condition is a natural one, since all Birkhoff interpolation problems occurring in practice satisfy it.

The general tenor of the answers is that if m and n are large, then almost no $m\times n$ incidence matrix is regular. Moreover, even if we restrict ourselves to more favourable subclasses, almost no matrices are regular.

The classes in question are $M(m, n)$, $P(m, n)$, $B(m, n)$, $M^*(m, n)$, $P^*(m, n)$ and $B^*(m, n)$, which are defined by:

a) $M(m, n)$ is the set of all $m\times n$ incidence matrices with exactly n ones among the entries.

b) $P(m, n)$ is the subset of $M(m, n)$, consisting of those matrices, which satisfy the Pólya condition $\sum_{j=0}^{k-1}\sum_{i=1}^{m} e_{ij}\geq k$, $k=1,\dots,n$, which says that the first k columns of E contain at least k ones.

c) $B(m, n)$ is the subset of $M(m, n)$ consisting of those matrices, satisfying the Birkhoff condition $\sum_{j=0}^{k-1}\sum_{i=1}^{m} e_{ij}\geq k+1$, $k=1,\dots,n$, which says that the first k columns of E contain at least $k+1$ ones.

d) $M^*(m, n)$ is the subset of $M(m, n)$ of those incidence matrices, having at least one 1 in each row. For these matrices, all knots really appear in system 1; they are all 'active'.

e) $P^*(m, n)=P(m, n)\cap M^*(m, n)$;

f) $B^*(m, n)=B(m, n)\cap M^*(m, n)$.

In [2] and [3] it was shown that if m and n become large, the probability that an incidence matrix from $M(m, n)$, $P(m, n)$ or $B(m, n)$ be regular, tends to zero.

We obtain the same conclusion for $M^*(m, n)$. Under the additional assumption that $m>\lambda n$ for some $\lambda>1/2$, this is also true for $B^*(m, n)$.

From the theory of Birkhoff interpolation it follows that an incidence matrix, which is regular, must necessarily satisfy the Pólya condition. Our approach is thus to show that there are relatively few Pólya matrices in $M^*(m, n)$.

To do this, we introduce the classes $M(m, n; n_1)$, $M^*(m, n; n_1)$, $P(m, n; n_1)$ and $P^*(m, n; n_1)$, which are as above except that each matrix has n_1 ones instead of n ones. We also make the convention that a class and the number of matrices in the class will be denoted by the same symbol, e.g.

$$M(m, n; n_1)=\binom{mn}{n_1} \tag{2}$$

(the binomial coefficient). This is just the number of different ways of placing n_1 ones in mn positions.

Unfortunately, it was not possible to find such a simple formula for $M^*(m, n; n_1)$, which makes the proofs here completely different from those in [2].

L e m m a 1. *For any m, n, n_1 with $m\leq n_1$, $M^*(m, n; n_1)$ is given by any of the following expressions:*

$$\sum_{i=1}^{m}(-1)^{m+i}\binom{m}{i}M(i, n; n_1); \tag{3}$$

(4) $$\sum_{i=1}^{n}(-1)^{m+1}\binom{m}{i}\binom{in}{n_1};$$

(5) $$\sum_{\substack{i_1,\ldots,i_m\geqq 1\\ \sum_{j=1}^{m} i_j=n_1}}\binom{n}{i_1}\cdots\binom{n}{i_m};$$

(6) $$\frac{1}{(n_1-m)!}\;\frac{d^{n_1-m}}{dy^{n_1-m}}\left[\frac{(1+y)^n-1}{y}\right]^m_{y=0};$$

(7) $$\frac{m!}{n_1!}\sum_{i=m}^{n_1} n^i S^i_{n_1}\mathsf{S}^m_i;$$

(8) $$\left[\Delta^m_n\binom{x}{n_1}\right]_{x=0},$$

where S^i_n *and* S^i_n *are the Stirling's numbers of the first second kinds respectively and* $\overset{m}{\underset{n}{\Delta}} f(x)$ *is the* m-th *divided difference of* f *with step size* n.

Formulas (6), (7) and (8) were only given to show the close connection of our counting arguments to the theory of finite differences as in [1, 5, 6]. Formula (6), for example, yields the generating function for $M^*(m, n; n_1)$. This is not at all surprising inasmuch as we are counting the number of ways of distributing n_1 ones into boxes under certain conditions.

Formula (1) is proved by first showing that $M(m, n; n_1) = \Sigma_{i=1}^m\binom{m}{i}M^*(i, n; n_1)$ and then using the binomial inversion theorem (also called Jordan's formula).

Lemma 2. *For all* m, n, n_1 *with* $m\leqq n_1$, $P^*(m, n)=(n+1)^{-1}M^*(m, n+1; n)$, $B^*(m, n)=(n-1)^{-1}M^*(m, n-1; n)$.

The lemma without stars was proved by Lorentz and Riemenschneider [3]. The proof also holds for our case.

Theorem 3. *For all* m, n *sufficiently large*, $P^*(m, n)/M^*(m, n)\leqq c/(n+1)$ *for a constant* c *independent of* m *and* n.

In view of Lemma 2 one only needs to show that $M^*(m, n+1; n)/M^*(m, n)$ remains bounded for large m and n. The proof of this fact is of combinatorial nature and is based on identity 5.

Corollary 4. *As* m, n *increase without bound, the probability that an incidence matrix from* $M^*(m, n)$ *be regular becomes vanishingly small.*

Since $M^*(m, n+1; n)>M^*(m, n)$, we have an upper and a lower bound for the proportion of Pólya matrices in $M^*(m, n)$, namely

$$1/(n+1)\leqq P^*(m, n)/M^*(m, n)\leqq c/(n+1).$$

Similar bounds hold for the proportion of Birkhoff matrices in $M^*(m, n)$:

(9) $$c/(n-1)\leqq B^*(m, n)/M^*(m, n)\leqq 1/(n-1).$$

These inequalities follow from Lemma 2, the fact that $M^*(m, n-1; n) < M^*(m, n)$ and the inequality $M^*(m, n)/M^*(m, n-1; n)\leqq c$ for a constant independent of m and n. Its proof is similar to that of the inequality of Theorem 3.

With the following lemma we will have the main tools for proving the last theorem.

Lemma 5. *Let E satisfy the Birkhoff condition and have three rows numbered* $i_1 < i_2 < i_3$, *which have the following properties:*

a) *row* i_2 *has exactly one* 1, *which is at position* (i_2, k);

b) *rows* i_1 *and* i_3 *have ones at position* (i_1, k_1) *and* (i_3, k_3) *respectively. Then, if* $k_2 > k_1$ *and* $k_2 > k_3$, *E is singular.*

This is a special case of a more general singularity theorem, given in [7].

Theorem 6. *Let* $m > \lambda n$ *for some* $\lambda > 1/2$. *Then as m and n become infinitely large, the proportion of regular matrices in* $B^*(m, n)$ *becomes vanishingly small.*

The idea of the proof is as follows. First, one shows that almost all matrices of $M^*(m, n)$ have the configuration described in Lemma 5. This is proved by noticing that if $m > \lambda n$, then at least $(\lambda - 1/2)n$ rows of each $E \in M^*(m, n)$ contain exactly one 1. By combinatorial arguments, one then shows that at most

$$c\left[\frac{12}{n^{1/3}}\right]^n M^*(m, n)$$

matrices do not have the desired configuration of ones. But by 9 we may conclude that, for large n, a vanishingly small proportion of $B^*(m, n)$ does not have the configuration of Lemma 5 and thus all but a vanishingly small proportion is singular.

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