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ON COTYPES OF BAHACH LATTICES

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Summary. A characterization of the cotypes of Banach lattices is given in terms of a new modulus of convexity γ_X. The dual to γ_X modulus σ_X is considered and exact estimates for γ_X and σ_X are found for Orlicz spaces. In terms of γ_X and σ_X a characterization of normed lattices, which are isomorphic or isometrically isomorphic to inner product space, is given.

1. Introduction. Figiel and Pisier [4] proved that every Banach space X is of type ρ_X and of cotype δ_X, where ρ_X and δ_X are the usual moduli of smoothness and convexity of X. We note that the result about type is nontrivial, when X is uniformly smooth, and about cotype — when X is uniformly convex. It is well-known (see [1]) that in X exists an equivalent uniformly smooth or uniformly convex norm iff X is superreflexive, so that the result of Figiel and Pisier give information about type and cotype of superreflexive Banach spaces. Recently in [3] for superreflexive Banach lattices was proved the following converse in some sense theorem: if X is of type (cotype) f, then there is in X an equivalent norm $|||\cdot|||$ such that $\rho_{(X, |||\cdot|||)}(\tau)\leqq c_1 f(\tau)$ $(\delta_{(X, |||\cdot|||)}(\varepsilon)\geqq c_2 f(\varepsilon))$.

Naturally the problem arises to find a characterization of cotypes of arbitrary Banach lattice. A first result in this direction was obtained by Tokarev [15], namely: the Banach lattice X does not contain uniformly l_∞^n for every n iff for some equivalent norm $|||\cdot|||$ in X the complex modulus of convexity of the standard complexification of the lattice $(X, |||\cdot|||)$ has a lower estimate of power type.

In this paper for Banach lattices a new modulus γ_X, analogous to the complex modulus of convexity, is introduced. A characterization of the cotypes of Banach lattices in terms of γ_X is obtained. A dual to γ_X modulus σ_X is considered and exact estimates for γ_X and σ_X are found, when X is an Orlicz space. From these estimates it follows that the spaces $L_M(0,1)$ (l_M) are of cotype p, $2\leqq p<\infty$, iff $M(t^{1/p})$ is quasiconcave in $[1, \infty)$ $([0, 1])$. A different proof of the 'if' part was given in [6].

We give also a characterization of inner product Banach lattices in terms of γ_X and σ_X. A part of these results was announced in [12].

2. Notations, Definitions. As usual X^* and S_X denote the conjugate space and unite sphere of X. For a complex Banach space X the function

$$\beta_X(\varepsilon)=\inf\{\sup\{\|x+\alpha\varepsilon y\|-1:|\alpha|=1\}:x,y\in S_X\},\quad \varepsilon\geq 0,$$

is called complex modulus of convexity of X.

If X is a Banach lattice, we shall call the functions

$$\gamma_X(\varepsilon)=\inf\{\|\sqrt{x^2+\varepsilon^2 y^2}\|-1:x,y\in S_X\},\quad \varepsilon\geq 0,$$
$$\sigma_X(\tau)=\sup\{\|\sqrt{x^2+\tau^2 y^2}\|-1:x,y\in S_X\},\quad \tau\geq 0,$$

order moduli of convexity and smoothness of X.

If $f:[0,a)\to[0,\infty)$, we say that the Banach space X is of type (cotype) f, if a constant $c<\infty$ exists, so that for every finite set $\{x_i\}_{i=1}^n$ of elements in X we have

$$\int_0^1\|\sum_{i=1}^n r_i(t)x_i\|\,dt\leq c\sum_{i=1}^n f(\|x_i\|);$$
$$\left(\int_0^1\|\sum_{i=1}^n r_i(t)x_i\|\,dt\geq c^{-1}\sum_{i=1}^n f(\|x_i\|)\right),$$

where $r_i(t), i-1, 2, \dots, n$, are the first n Rademacher functions.

The functions f and g will be called equivalent at 0, if there are positive constants c_1, c_2, c_3, c_4, c_5 such that

$$c_1 f(c_2 x)\leq g(x)\leq c_3 f(c_4 x),\quad x\in[0,c_5].$$

If X is a Banach space, then the space of all X-valued functions $x(t)$ defined on $[0,1]$ with the norm $\|x\|=(\int_0^1\|x(t)\|^2\,dt)^{1/2}$ is denoted $L_2(X)$. With $c_1, c_2, c_3, \dots$ we denote always positive constants.

3. Characterization of the Cotype of Banach Lattices. Everywhere in this section, unless it is otherwise specified, X is a Banach lattice. For the proof of the characterization theorem we need some lemmas.

Lemma 3.1. *If a, b, c, d are arbitrary real numbers, the following inequality holds:*

$$(1)\qquad \sqrt{(a+c)^2+(b+d)^2}+\sqrt{(a-c)^2+(b-d)^2}+\sqrt{(a-d)^2+(b+c)^2}+\sqrt{(a+d)^2+(b-c)^2}\geq\sqrt{16(a^2+b^2)+2(c^2+d^2)}.$$

Proof. One can use the inequality [5]

$$|\rho e^{i\alpha}+1|+|\rho e^{i\alpha}-1|+|\rho e^{i\alpha}+i|+|\rho e^{i\alpha}-i|\geq\sqrt{16\rho^2+2}$$

for arbitrary α and $\rho\geq 0$. If we take $\rho=\lambda/\mu$ and $\alpha=\varphi-\psi$, then for $\xi=\lambda e^{i\varphi}$, $\eta=\mu e^{i\psi}$ we obtain the inequality

$$|\xi+\eta|+|\xi-\eta|+|\xi+i\eta|+|\xi-i\eta|\geq\sqrt{16|\xi|^2+2|\eta|^2}.$$

To get (1) it is enough to put in this inequality $\xi=a+ib$, $\eta=c+id$.

Corollary 3.2. *For every $u, v, s, t\in X$ holds the inequality*

$$\|\sqrt{(u+s)^2+(v+t)^2}+\sqrt{(u-s)^2+(v-t)^2}+\sqrt{(u-t)^2+(v+s)^2}+\sqrt{(u+t)^2+(v-s)^2}\|\geq 4\|\sqrt{u^2+v^2+(s^2+t^2)/8}\|$$

Corollary 3.3. If $\tilde{X}$ *is the complexification of* X *(see e. g.* [10, p.43]), *then* $\gamma_X(\varepsilon/3)\leq\beta_{\tilde{X}}(\varepsilon)$.

This result follows easily from Corollary 3.2 and the fact that

$$\beta_{\tilde{X}}(\varepsilon)=\inf\{\max(\|x\pm\varepsilon y)\|, \|x\pm i\varepsilon y\|): x, y\in S_{\tilde{X}}\}-1.$$

Proposition 3.4 (complex equivalent of the Kadec's theorem [8], see [14]). If X is a complex Banach space, then for every finite set $\{x_i\}_{i=1}^n$ of elements in X with $\max\{\|\Sigma_{i=1}^n \varepsilon_i x_i\|:|\varepsilon_i|=1\}\leq 1$ the inequality holds $\Sigma_{i=1}^n \beta_X(\|x_i\|)\leq 2$.

Proof. Let the set $\{x_i\}_{i=1}^n$ in X satisfies the condition $\max\{\|\Sigma_{i=1}^n \varepsilon_i x_i\|:|\varepsilon_i|=1\}\leq 1$. Without loss of generality we shall assume that the maximum is attained for $x=\Sigma_{i=1}^n x_i$. Let $x^*\in X^*$ with $x^*(x)=\|x\|$. Denote $z_i=x-x_i$. Obviously $\|z_i\|\leq\|x\|\leq 1$. Now we shall prove that

$$\beta_X(\|x_i\|)\leq\frac{2}{\|x\|}|x^*(x_i)|, \quad i=1, 2, \ldots, n. \tag{2}$$

If $\|z_i\|\geq\|x\|/2$, then

$$\beta_X(\|x_i\|)\leq\beta_X(\|x_i\|/\|z_i\|)\leq\max\{\|z_i/\|z_i\|+\alpha x_i/\|z_i\|:|\alpha|=1\}-1$$

$$=\frac{\|x\|-\|z_i\|}{\|z_i\|}\leq\frac{2}{\|x\|}(x^*(x)-\|z_i\|)\leq\frac{2}{\|x\|}(|x^*(x_i)|+|x^*(z_i)|-\|z_i\|)$$

$$\leq\frac{2}{\|x\|}|x^*(x_i)|.$$

If $\|z_i\|<\|x\|/2$, then $|x^*(x_i)|\geq|x^*(x)|-|x^*(z_i)|>\|x\|/2$ and $\beta_X(\|x_i\|)\leq\beta_X(\|x\|)\leq\|x\|\leq 2|x^*(x_i)|\leq(2/\|x\|)|x^*(x_i)|$.

Thus (2) is proved. Let now $|\alpha_i|=1$ and $x^*(\alpha_i x_i)=|x^*(x_i)|$. From (2) we obtain

$$\sum_{i=1}^n \beta_X(\|x_i\|)\leq\frac{2}{\|x\|}\sum_{i=1}^n x^*(\alpha_i x_i)=\frac{2}{\|x\|}x^*(\sum_{i=1}^n \alpha_i x_i)\leq\frac{2}{\|x\|}\|\sum_{i=1}^n \alpha_i x_i\|\leq 2,$$

which completes the proof.

Proposition 3.5. For every finite set $\{x_i\}_{i=1}^n$ of elements in X such that $\max\{\|\Sigma_{i=1}^n \varepsilon_i x_i\|:\varepsilon_i=\pm 1\}\leq 1/6$ the inequality $\Sigma_{i=1}^n \gamma_X(\|x_i\|)\leq 2$ holds

Proof. Suppose that $\max\{\|\Sigma_{i=1}^n \varepsilon_i x_i\|:\varepsilon_i=\pm 1\}\leq 1/6$ and let $\tilde{x}_i=(x_i, 0)$ are elements of the standard complexification $\tilde{X}$ of X. Obviously

$$\max\{\|\sum_{i=1}^n \varepsilon_i \tilde{x}_i\|_{\tilde{X}}:|\varepsilon_i|=1\}\leq 2\max\{\|\sum_{i=1}^n \varepsilon_i \tilde{x}_i\|_{\tilde{X}}:\varepsilon_i=\pm 1\}$$

$$\leq 2\max\{\|\sum_{i=1}^n \varepsilon_i x_i\|_X:\varepsilon_i=\pm 1\}\leq 1/3.$$

Now by using Corollary 3.3 and Proposition 3.4 we obtain

$$\sum_{i=1}^n \gamma_X(\|x_i\|)\leq\sum_{i=1}^n \beta_{\tilde{X}}(3\|x_i\|)\leq 2.$$

Lemma 3.6. *Let X, Y be normed lattices and $T: X\to Y$ be an order isomorphism with $\overline{TX}=Y$. If $x\in X$, $y\in Y$ and $x\geq 0$, $0\leq y\leq Tx$, then for every $\delta>0$ it exists $z\in X$ such that*

$$0\leq z\leq x, \quad \|y-Tz\|<\delta.$$

Proof. Let $\delta>0$. Pick $v\in X$, so that $\|u-y\|<\delta$, where $u=Tv$. Then we have for $z=0\vee u\wedge x$

$$Tz=T0\vee Tv\wedge Tx=0\vee(u\wedge Tx)=(u\wedge Tx)^{+}=(u\wedge Tx+|u\wedge Tx|)/2$$
$$=(u+Tx-|u-Tx|+|u+Tx-|u-Tx||)/4$$

and for $y: y=(y\wedge Tx)^{+}=(y+Tx-|y-Tx|+|y+Tx-|y-Tx||)/4$.

From these representations it follows $|y-Tz|\leq(|y-u|+|y-u|+|y-u+|u-Tx|-|y-Tx||)/4\leq|y-u|$. Thus $\|y-Tz\|\leq\|y-u\|<\delta$ and our assertion is proved.

Lemma 3.7 (see e. g. [10, p.43]). *For every $x, y\in X$ and $x^*, y^*\in X^*$ denote $z=\sqrt{x^2+y^2}$, $z^*=\sqrt{x^{*2}+y^{*2}}$. Then $z^*(z)\geq x^*(x)+y^*(y)$.*

Lemma 3.8. *Let $z\in X$, $x^*, y^*\in X^*$ and $x^*, y^*, z\geq 0$. If $z^*=\sqrt{x^{*2}+y^{*2}}$, then for every $\delta>0$ there exist $x, y\in X$ such that $x, y\geq 0$, $z=\sqrt{x^2+y^2}$, $\|x\|\geq x^*(z)/(2\|z^*\|)$ and $z^*(z)\leq x^*(x)+y^*(y)+\delta$.*

Proof. Let $0<\delta<1$. We assume that $x^*(z)>0$ (if $x^*(z)=0$ one can take $x=0$, $y=z$). From a theorem of Kakutani (see e. g. [10, p.44]) there exist a measure space (Ω,Σ,μ) and an isometric order isomorphism $T:X\to L_1(\Omega,\Sigma,\mu)$, so that $\|T\|=\|z^*\|$ and $\overline{TX}=L_1(\Omega,\Sigma,\mu)$. Then we can find $f, g\in L_\infty(\Omega,\Sigma,\mu)$, $f, g\geq 0$ such that $\|f\|_\infty=\|g\|_\infty\leq 1$ and $x^*(u)=\int_\Omega Tuf\,d\mu$, $y^*(u)=\int_\Omega Tug\,d\mu$ for every $u\in X$. Obviously, if $h=\sqrt{f^2+g^2}$, then $z^*=\sqrt{x^{*2}+y^{*2}}=T^*h$.

Let $\xi=Tzf/h$, $\eta=Tzg/h$. From $\|h\|\leq\sqrt{2}$ it follows

$$\|\xi\|\geq\int_\Omega Tzfd\mu/\sqrt{2}\geq x^*(z)/\sqrt{2}. \tag{3}$$

We choose $\alpha>0$ with $\alpha<\min\{((\sqrt{2}-1)/2)x^*(z),\ \delta/(2+4\|\xi\|)\}$. The inequalities $0\leq\xi\leq Tz$ and Lemma 3.6 imply the existence of an $x\in X$ with $0\leq x\leq z$, $\|Tx-\xi\|<\alpha$. Then from (3) we deduce $\|x\|\geq\|Tx\|/\|z^*\|\geq(\|\xi\|-\alpha)/\|z^*\|\geq x^*(z)/(2\|z^*\|)$.

Put now $y=\sqrt{|z^2-x^2|}$. Clearly $z=\sqrt{x^2+y^2}$ and $Tz=\sqrt{(Tx)^2+(Ty)^2}$. We have $\|Tx-\xi\|<\delta/2$. The same estimate is true for $\|Ty-\eta\|$. Indeed

$$\|Ty-\eta\|=\|\sqrt{(Tz)^2-(Tx)^2}-\eta\|=\|\sqrt{\xi^2+\eta^2-(Tx)^2}-\eta\|$$
$$\leq\|\sqrt{\xi^2-(Tx)^2}\|=\|\sqrt{|\xi-Tx|(\xi+Tx)}\|\leq\|\xi-Tx\|\,\|\xi+Tx\|$$
$$\leq\|\xi-Tx\|(\|\xi-Tx\|+2\|\xi\|)\leq\alpha(1+2\|\xi\|)<\delta/2.$$

On the other hand, $\int_\Omega(\xi f+\eta g)\,d\mu=\int_\Omega Tzh\,d\mu=z^*(z)$ and

$$|x^*(x)-\int_\Omega\xi fd\mu|\leq\int_\Omega|Txf-\xi f|\,d\mu\leq\|f\|\,\|Tx-\xi\|\leq\delta/2,$$

$$|y^*(y)-\int_\Omega\eta gd\mu|\leq\delta/2.$$

This completes the proof.

Lemma 3.9. *The following duality formula holds*

$$\sigma_{X^*}(\tau)=\sup\left\{\frac{\varepsilon\tau-\gamma_X(\varepsilon)}{1+\gamma_X(\varepsilon)}:\varepsilon\geq 0\right\},\quad \tau\geq 0.$$

Proof. We prove first that for every $\varepsilon, \tau \geqq 0$

(4) $$(1+\gamma_X(\varepsilon))\,(1+\sigma_{X^*}(\tau)) \geqq 1+\varepsilon\tau.$$

Let $\delta>0$, $x, y \in S_X$ with $1+\gamma_X(\varepsilon) > \| \sqrt{x^2+\varepsilon^2 y^2} \| - \delta$ and $x^*, y^* \in S_{X^*}$ with $x^*(x)=y^*(y)=1$. If $z=\sqrt{x^2+\varepsilon^2 y^2}$, $z^*=\sqrt{x^{*2}+\tau^2 y^{*2}}$, Lemma 3.7 implies

$$1+\varepsilon\tau \leqq x^*(x)+\tau\, y^*(\varepsilon y) \leqq z^*(z) \leqq \| z^* \|\, \| z \| < (1+\sigma_{X^*}(\tau))\,(1+\gamma_X(\varepsilon)+\delta).$$

Thus (4) is proved.

Let now $0<\eta<1/(8\sqrt{1+\tau^2})$. There exist $x^*, y^* \in S_{X^*}$, $x^*, y^* \geqq 0$ such that $1+\sigma_{X^*}(\tau) \leqq \| z^* \| + \eta$, $z^*=\sqrt{x^{*2}+\tau^2 y^{*2}}$ and $z \in S_X$ with $z \geqq 0$, $z^*(z) \geqq \| z^* \| - \eta$. Obviously $x^*(z) \geqq z^*(z) - \| z^* - x^* \|$. Then $\| z^* - x^* \| \leqq \| \tau y^* \| = \tau$ implies $x^*(z) \geqq \| z \| - \tau - \eta$. According to Lemma 3.8, there are $u, v \in X$, $u, v \geqq 0$ such that $z=\sqrt{n^2+v^2}$ and

(5) $$\| u \| \geqq x^*(z)/(2\| z^* \|) \geqq (\| z^* \| - \eta - \tau)/(2\| z^* \|) = \frac{1}{2} - (\eta+\tau)/(2\| z^* \|),$$

$$z^*(z) \leqq x^*(u) + \tau\, y^*(v) + \eta.$$

But $\sigma_{X^*}(\tau) \geqq \sqrt{1+\tau^2}-1$. Therefore $\| z^* \| \geqq \sqrt{1+\tau^2}-\eta$ and (5) imply $\| u \| > 1/(8\tau^2+8)$. Put now $x=\lambda u$, $y=\lambda v$ with $\lambda = 1/\| u \|$. We obtain $(1+\gamma_X(\| y \|))\,(1+\sigma_{X^*}(\tau)) \leqq \lambda(z^*(z)+\eta) \leqq x^*(x)+\tau y^*(y)+2\lambda\eta \leqq 1+\tau \| y \| + 2\lambda\eta$.

From $\lambda < 8(\tau^2+1)$ it follows

$$\sigma_{X^*}(\tau) \leqq \sup\{ \frac{\varepsilon\tau-\gamma_X(\varepsilon)}{1+\gamma_X(\varepsilon)} : \varepsilon \geqq 0\} + 16\eta(1+\tau^2),$$

which completes the proof.

In the sequel we shall denote with $\tilde{\varphi}$ the maximal convex function minorizing φ, i. e. for which $\tilde{\varphi} \leqq \varphi$.

Lemma 3.10 [11]. *Let φ satisfy the condition $1+\varphi(t)>0$ for $t \geqq 0$. If $A\varphi=\psi$ is the operator defined with $\psi(\tau)=\sup\{(t\tau-\varphi(t))/(1+\varphi(t)) : t \geqq 0\}$, then $A^2\varphi=\tilde{\varphi}$.*

Lemma 3.11. *Let $\alpha_X(\varepsilon)=\gamma_X(\sqrt{\varepsilon})$. Then $\alpha_X \sim \tilde{\alpha}_X$.*

Proof. It is easy to check that $\sigma_X(2\tau) \leqq 4\sigma_X(\tau)$ and one can find as in [2] a constant $c_1>0$ such that for $\tau_2 \leqq \tau_1$, $\tau_1, \tau_2 \in (0, 1]$

$$\frac{\sigma_{X^*}(\tau_1)}{\tau_1^2} \leqq c_1 \frac{\sigma_{X^*}(\tau_2)}{\tau_2^2}.$$

Then from Lemma 3.9 and Lemma 3.10 it follows that

$$\frac{\tilde{\gamma}_X(\varepsilon_1)}{\varepsilon_1^2} \leqq c_2 \frac{\tilde{\gamma}_X(\varepsilon_2)}{\varepsilon_2^2} \quad \text{for } \varepsilon_1 \leqq \varepsilon_2,\ \varepsilon_1, \varepsilon_2 \in (0, 1],$$

where c_2 is a positive constant. On the other hand, $\gamma_X(\varepsilon)/\varepsilon$ is nondecreasing as infimum of nondecreasing functions. Therefore (see e. g. [2]) $\gamma_X \sim \tilde{\gamma}_X$, so that for some $c_3>0$ and all $\varepsilon_1, \varepsilon_2 \in (0,1]$, $\varepsilon_1 \leq \varepsilon_2$ we have $\alpha_X(\varepsilon_1)/\varepsilon_1 \leq c_3\, \alpha_X(\varepsilon_2)/\varepsilon_2$. The same argument shows that $\alpha_X \sim \tilde{\alpha}_X$.

Lemma 3.12. *The order moduli γ_X and $\gamma_{L_2(X)}$ are equivalent at zero, i. e. $\gamma_X \sim \gamma_{L_2(X)}$.*

Proof. Let $\varphi, \psi \in S_{L_2(X)}$, $B=\{t \in [0,1] : \varphi(t) \neq 0\}$, $C=[0,1] \setminus B$. Then for $\varepsilon \in [0,1]$

$$\|\sqrt{\varphi^2+\varepsilon^2\psi^2}\|^2_{L_2(X)} = \int_0^1 \|\sqrt{\varphi^2(t)+\varepsilon^2\psi^2(t)}\|^2\, dt$$

$$\geq \varepsilon^2 \int_C \|\psi(t)\|^2\, dt + \int_B \|\varphi(t)\|^2 (\gamma_X(\varepsilon\|\psi(t)\|/\|\varphi(t)\|)+1)^2\, dt$$

$$\geq 1+\varepsilon^2 \int_C \|\psi(t)\|^2\, dt + \int_B \|\varphi(t)\|^2\, \gamma_X(\varepsilon\|\psi(t)\|/\|\varphi(t)\|)\, dt$$

$$\geq 1+\tilde{\alpha}_X(\varepsilon^2 \int_B \|\psi(t)\|^2\, dt)+\tilde{\alpha}_X(\varepsilon^2 \int_C \|\psi(t)\|^2\, dt) \geq 1+2\tilde{\alpha}_X(\varepsilon^2/2).$$

Now from Lemma 3.10 we obtain

$$\|\sqrt{\varphi^2+\varepsilon^2\psi^2}\|_{L_2(X)}-1 \geq \frac{2}{3}\tilde{\alpha}_X(\varepsilon^2/2) \geq c_1\, \gamma_X(c_2\, \varepsilon).$$

Thus the lemma is proved.

Theorem 3.13. *A Banach lattice X is of cotype f iff in X it exists an equivalent norm $|||\cdot|||$, so that*

$$\gamma_{(X,|||\cdot|||)}(\varepsilon) \geq c_1 f(\varepsilon),\ \ \varepsilon \in [0, c_2]. \tag{6}$$

Proof. The 'if' part results from Lemma 3.11 and Proposition 3.5. If X is of cotype f, an equivalent norm, satisfying (6), is constructed in [3]

Remark 3.14. We point out that the theorem is nontrivial only if X does not contain uniformly l^n_∞ for every n. If this condition is not fulfilled, γ_X and the cotype f of X are identically zero in a neighbourhood of 0.

Corollary 3.15. *Let $\{e_i\}_{i=1}^\infty$ be unconditional normalized basic sequence in X and $\{u_i\}_{i=1}^\infty$ an ω-linear independent system in X. If*

$$\sum_{i=1}^\infty \sigma_{X^*}(\|u_i-e_i\|)<\infty,$$

then $\{u_i\}_{i=1}^\infty$ is unconditional basic sequence in X.

This result generalizes a stability theorem for unconditional basis from [7] and can be proved with the same method by using Lemma 3.8.

4. Estimates for γ_X and σ_X in Orlicz Spaces. Everywhere in this section M denotes Orlicz function. Let us recall that M is said to have the property Δ_2 at 0 and ∞ (at 0, at ∞), if

$$M(2x) \leq kM(x),\ x \in [0,\infty),\ (x \in [0, 1],\ x \in [1,\infty)), \tag{7}$$

where $k<\infty$ is a positive constant.

An equivalent condition (see e. g. [9, p.37]) to (7) is the following

$$\frac{xM'(x)}{M(x)} \leq k < \infty,\ x \in (0,\infty)\ (x \in (0,1],\ x \in [1,\infty)). \tag{8}$$

The Orlicz function N: $N(x)=\sup\{xt-M(t):t\geqq 0\}$ is called complementary to M.

We shall consider Orlicz spaces $L_M(S, \Sigma, \mu)$ over measure spaces (S, Σ, μ) of the following types:

(A) $\mu(S)=\infty$ and there is $R\in\Sigma$, $\mu(R)>0$, R free of atoms;

(B) $\mu(S)=\infty$ and μ is purely atomic with atoms of equal mass;

(C) $\mu(S)<\infty$ and μ is not purely atomic.

The most important Orlicz spaces $L_M(0, \infty)$, l_M, $L_M(0, 1)$ are of type (A), (B), (C), respectively.

We introduce the following functions necessary for the estimation of γ_{L_M} and σ_{L_M}:

$$F_{M,I}(\varepsilon)=\inf\{(\frac{\varepsilon}{u})^2\frac{M(uv)}{M(v)}:u\in[\varepsilon, 1], v\in I\}, \quad \varepsilon\in[0, 1];$$

$$G_{M,I}(\tau)=\sup\{(\frac{\tau}{u})^2\frac{M(uv)}{M(v)}:u\in[\tau, 1], v\in I\}, \quad \tau\in[0, 1];$$

$$\mathscr{F}_{M,I}(\varepsilon)=\inf\{(\varepsilon u)^2\frac{M(v)}{M(uv)}:u\in[1, 1/\varepsilon], v\in I\}, \quad \varepsilon\in[0, 1];$$

$$\mathscr{G}_{M,I}(\tau)=\sup\{(\tau u)^2\frac{M(v)}{M(uv)}:u\in[1, 1/\tau], v\in I\}, \quad \tau\in[0, 1].$$

We put $F_{M,I}(\varepsilon)=0$, $G_{M,I}(\tau)=\tau^2$ and $\mathscr{F}_{M,I}(\varepsilon)=0$, $\mathscr{G}_{M,I}(\tau)=\tau^2$, if I contains an interval, where M is identically zero.

Let us note that, if M has the property Δ at 0, $F_{M,(0,1]}\sim F_{M,(0,c]}$, $G_{M,(0,1]}\sim G_{M,(0,c]}$, and if M has the property Δ_2 at ∞, then $\mathscr{F}_{M,[1,\infty)}\sim\mathscr{F}_{M,[c,\infty)}$, $\mathscr{G}_{M,[1,\infty)}\sim\mathscr{G}_{M,[c,\infty)}$ for every $c>0$. It is not difficult to see that if M has not the property Δ_2 at 0 and ∞ (at 0, at ∞), then the function $F_{M,(0,\infty)}$ ($F_{M,(0,1]}$, $\mathscr{F}_{M,[1,\infty)}$) is identically zero on (0, 1).

Lemma 4.1. *Let M, N be complementary Orlicz functions. Then for* $0<\tau\leqq\varepsilon\leqq 1$

(9) $$F_{N,(0,\infty)}(\varepsilon)\geqq\varepsilon\tau-G_{M,(0,\infty)}(\tau).$$

Proof. If $\varepsilon\tau\leqq G_{M,(0,\infty)}(\tau)$, (9) is obvious. Suppose $G_{M,(0,\infty)}(\tau)<\varepsilon\tau$. Pick $u\in[\varepsilon, 1]$ and $v>0$. We can assume that $N(v)>0$ (if N is identically zero on $[0, v]$, then $F_{N,(0,\infty)}=0$, $G_{N,(0,\infty)}(\tau)=\tau^2$ by definition and (9) is satisfied). Now we fix arbitrary $\eta\in(0, N(v))$ and w_0 with $N(v)<vw_0-M(w_0)+\eta$. Evidently

(10) $$vw_0-M(w_0)>N(v)-\eta>0.$$

It is easy to check that if $t_0=u\tau/\varepsilon$, then $\tau\leqq t\leqq 1$ and $\varepsilon\tau>\tau^2 M(t_0 w_0)/(t_0^2 M(w_0))=\varepsilon^2 M(t_0 w_0)/(u^2 M(w_0))$. So we have

(11) $$u^2\tau M(w_0)\geqq\varepsilon M(t_0 w_0).$$

On the other hand, $N(uv)\geqq uvt_0w_0-M(t_0w_0)$ and from (11) it follows

(12) $$\varepsilon^2 N(uv)\geqq u^2\varepsilon\tau(vw_0-M(w_0)).$$

Using (10), we obtain from (12)

$$\varepsilon^2 N(uv) \geqq u^2 \varepsilon\tau(vw_0 - M(w_0)) - \frac{u^2\tau^2}{t_0^2} \frac{M(t_0 w_0)}{M(w_0)} (vw_0 - M(w_0))$$

$$\geqq u^2 \left(\varepsilon\tau - \left(\frac{\tau}{t_0}\right)^2 \frac{M(t_0 w_0)}{M(w_0)}\right)(vw_0 - M(w_0)) \geqq u^2(\varepsilon\tau - G_{M,(0,\infty)}(\tau))(N(v) - \eta).$$

To complete the proof it is enough to put $\eta \to 0$ in the last inequality.

Remark 4.2. It is well-known that if p and q are the right derivatives of M and N respectively, then $N(v) = vq(v) - M(q(v))$ and p is inverse to q. So if $v \in [0, c = p(1/2)]$ ($v \in [d = p(2), \infty)$), then w_0 in the inequality (10) can be chosen in the interval $(0, 1]$ ($[1, \infty)$) and by using the note before Lemma 4.1 and repeating step by step the proof of Lemma 4.1, one obtains for $0 \leqq \tau \leqq \varepsilon \leqq 1$

$$F_{N,(0,c]}(\varepsilon) \geqq \varepsilon\tau - G_{M,(0,1]}(\tau), \tag{13}$$

$$\mathscr{F}_{N,[d,\infty)}(\varepsilon) \geqq \varepsilon\tau - \mathscr{S}_{M,[1,\infty)}(\tau). \tag{14}$$

Remark 4.3. As we noted above, if M does not satisfy the property Δ_2 at 0 and ∞ (at 0, at ∞), then $F_{M,(0,\infty)}$ ($F_{M,(0,1]}$, $\mathscr{F}_{M,[1,\infty)}$) is identically zero on (0, 1) and from Lemma 4.1 (from (13), from (14)) it follows that in this case $G_{N,(0,\infty)}(\tau) \geqq \tau$ ($G_{N,(0,1]}(\tau) \geqq \tau$, $\mathscr{S}_{N,[1,\infty)}(\tau) \geqq \tau$) for $\tau \in [0, 1]$.

Lemma 4.4. *Let* $a, b \geqq 0$ *and* $c = \max(a, b)$. *Then*

$$M(\sqrt{a^2+b^2}) - M(a) \geqq \frac{1}{9} \frac{b^2}{c^2} M(c).$$

Proof. We shall consider separately the cases:

a) $a \geqq b$: $$M(\sqrt{a^2+b^2}) - M(a) \geqq \frac{\sqrt{a^2+b^2}-a}{a} M(a) = \frac{b^2}{a+\sqrt{a^2+b^2}} M(a)$$

$$\geqq \frac{1}{3} \frac{b^2}{a^2} M(a) = \frac{1}{3} \frac{b^2}{c^2} M(c);$$

b) $\frac{b}{2} \leqq a < b$: $$M(\sqrt{a^2+b^2}) - M(a) \geqq M(\sqrt{(5/4)\, b^2}) - M(b)$$

$$= \left(\frac{\sqrt{5}}{2} - 1\right) M(b) \geqq \frac{1}{9} \frac{b^2}{c^2} M(c);$$

c) $a < \frac{b}{2}$: $$M(\sqrt{a^2+b^2}) - M(a) \geqq M(b) - M(b/2) \geqq \frac{1}{2} M(b) = \frac{1}{2} \frac{b^2}{c^2} M(c).$$

Lemma 4.5 ([12]). *For* $u \in [0, \infty)$, $v \in [0, 1]$, $t \in (0, 1]$:

$$M\left(\frac{uv}{t}\right) \leqq \frac{v^2 M(u)}{F_{M,(0,\infty)}(t)} + M(u).$$

Remark 4.6. In the same manner one can obtain that

$$M\left(\frac{uv}{t}\right) \leqq \frac{v^2 M(u)}{F_{M,(0,1]}(t)} + M(u) \tag{15}$$

for all triplets (u, v, t) with $u \in [0, \alpha]$, $v \in [0, 1]$, $t \in (0, 1]$ and $v/t \leq 4$, where $\alpha = M^{-1}(1)/4$, and

$$M\left(\frac{uv}{t}\right) \leq \frac{v^2 M(u)}{\mathcal{F}_{M,[c,\infty)}(t)} + M(u) \tag{16}$$

for every $c > 0$ and for all triplets (u, v, t) such that $u \in [c, \infty)$, $u \in [0, 1]$, $t \in (0, 1]$.

Theorem 4.7. *Let $X = L_M(S, \Sigma, \mu)$. The following estimates for γ_X hold:*

(i) *if* (S, Σ, μ) *is of type* (A)

$$\gamma_X(\varepsilon) \geq C_M F_{M,(0,\infty)}(\varepsilon), \quad \varepsilon \in [0, 1];$$

(ii) *if* (S, Σ, μ) *is of type* (B)

$$\gamma_X(\varepsilon) \geq C_M F_{M,(0,1]}(\varepsilon), \quad \varepsilon \in [0, 1];$$

(iii) *if* (S, Σ, μ) *is of type* (C)

$$\gamma_X(\varepsilon) \geq C_M F_{M,[1,\infty)}(\varepsilon), \quad \varepsilon \in [0, 1],$$

where the constant $C_M > 0$ depends only on M.

Proof. We note first that (i) ((ii), (iii)) trivially hold, if M has not the property Δ_2 at 0 and ∞, because in this case the function on the right-hand side of the inequality is identically zero on $[0, 1)$.

Let now (S, Σ, μ) be of type (A) and M has the property Δ_2 at 0 and ∞. For $f, g \in S_X$ we denote $h(x) = \max(|f(x)|, |g(x)|)$.

$$\int_S M(h(x)/2)\,\mu(dx) \leq \int_S M((|f(x)| + |g(x)|)/2)\,\mu(dx)$$

$$\leq \frac{1}{2} \int_S (M(|f(x)|) + M(|g(x)|))\,\mu(dx) \leq 1.$$

Thus $\|h\| \leq 2$. Applying Lemma 4.4 with $a = \sqrt{f^2(x) + \varepsilon^2 g^2(x)}$, $b = |f(x)|$ and the inequality from Lemma 4.5 with $u = h(x)/2$, $v = \varepsilon |g(x)| / h(x)$, $t = \varepsilon/4$, we get the inequalities

$$\frac{\varepsilon^2}{9} \frac{g^2(x)}{h^2(x)} M(h(x)) \leq M(\sqrt{f^2(x) + \varepsilon^2 g^2(x)}) - M(|f(x)|), \tag{17}$$

$$F_{M,(0,\infty)}(\varepsilon/4)\,(M(2|g(x)|) - M(h(x)/2)) \leq \varepsilon^2 \frac{g^2(x)}{h^2(x)} M(h(x)). \tag{18}$$

Combining (17) and (18) after integration on S, we obtain

$$\frac{1}{9} F_{M,(0,\infty)}(\varepsilon/4) \int_S (M(2|g|) - M(h/2))\,\mu(dx) \leq \int_S (M(\sqrt{f^2 + \varepsilon^2 g^2}) - M(|f|))\mu(dx).$$

But $\int_S M(h/2)\,\mu(dx) \leq 1$, $\int_S M(2|g|)\,\mu(dx) \geq 2 \int_S M(|g|)\,\mu(dx) \geq 2$.

Therefore $(1/9) F_{M,(0,\infty)}(\varepsilon/4) \leq \int_S M(\sqrt{f^2 + \varepsilon^2 g^2})\,\mu(dx) - 1$.

To estimate $\int_S M(\sqrt{f^2 + \varepsilon^2 g^2})\,\mu(dx)$ it is enough to use that M has the property Δ_2 at 0 and ∞. From (8) it follows

$$\int_S M(\sqrt{f^2+\varepsilon^2 g^2})\,\mu(dx)$$

$$\leqq \|\sqrt{f^2+\varepsilon^2 g^2}\|^k \int_S M\left(\frac{\sqrt{f^2+\varepsilon^2 g^2}}{\|\sqrt{f^2+\varepsilon^2 g^2}\|}\right)\mu(dx)=\|\sqrt{f^2+\varepsilon^2 g^2}\|^k.$$

Now we obtain

$$\|\sqrt{f^2+\varepsilon^2 g^2}\|-1\geqq\left(1+\frac{1}{9}F_{M,(0,\infty)}(\varepsilon/4)\right)^{1/k}-1$$

$$\geqq\frac{1}{18k}F_{M,(0,\infty)}(\varepsilon/4)\geqq C_M\,F_{M,(0,\infty)}(\varepsilon).$$

Thus (i) is proved. The proof of (ii) and (iii) by using the inequalities (15) and (16) from Remark 4.6 instead of Lemma 4.5 involves a little more computations, but is completely similar to that of (i) and we omit it.

Theorem 4.7. *Let* $X=L_M(S,\Sigma,\mu)$. *The following estimates for* σ_X *hold*:

(i) *if* (S,Σ,μ) *is of type* (A), $\sigma_X(\tau)\leqq D_M\,G_{M,(0,\infty)}(\tau)$, $0\leqq\tau\leqq 1$;
(ii) *if* (S,Σ,μ) *is of type* (B), $\sigma_X(\tau)\leqq D_M\,G_{M,(0,1]}(\tau)$, $0\leqq\tau\leqq 1$;
(iii) *if* (S,Σ,μ) *is of type* (C), $\sigma_X(\tau)\leqq D_M\,\mathcal{S}_{M,[1,\infty)}(\tau)$,

where $D_M>0$ *is constant, depending only on* M.

Proof. We begin with the proof of (i). Denote $A_M=\inf\{xM'(x)/M(x);\ x\in(0,\infty)\}$. Suppose first that $A_M=1$. It is well-known (see e. g. [9, p. 39]) that in this case the Orlicz function N complementary to M has not the property Δ_2 at 0 and ∞ and according to Remark 4.3 $G_{M,(0,\infty)}(\tau)\geqq\tau$. On the other hand, $\sigma_X(\tau)\leqq\tau$, so that $\sigma_X(\tau)\leqq G_{M,(0,\infty)}(\tau)$.

Let now $A_M>1$. The complementary function N of M has the property Δ_2 at 0 and ∞. Thus we have for $Y=L_N(S,\Sigma,\mu)$ $Y^*=X$ and $F_N(\varepsilon)>0$ for every $\varepsilon>0$. Therefore from Theorem 4.6 it follows $\gamma_Y(1)>0$. Let $K_N=\min(1,C_N)$. For every $\tau\in(0,\gamma_Y(1))$, $\varepsilon\geqq 1$ $K_N\varepsilon\tau-\gamma_Y(1)<\varepsilon\gamma_Y(1)-\gamma_Y(\varepsilon)\leqq 0$, because $\gamma_Y(\varepsilon)/\varepsilon$ is nondecreasing.

Thus for $\tau\in(0,\gamma_Y(1))$ $\sigma_X(K_N\tau)=\sup_{0\leqq\varepsilon<1}(K_N\varepsilon\tau-\gamma_Y(\varepsilon))/(1+\gamma_Y(\varepsilon))$ and we can estimate $\sigma_X(K_N\tau)$ in the following way:
$\sigma_X(K_N\tau)\leqq\sup_{0\leqq\varepsilon\leqq\tau}\{K_N\varepsilon\tau-\gamma_Y(\varepsilon)\}+\sup_{\tau\leqq\varepsilon<1}\{K_N\varepsilon\tau-\gamma_Y(\varepsilon)\}\leqq K_N\tau^2+\sup_{\tau\leqq\varepsilon<1}\{K_N\varepsilon\tau-\gamma_Y(\varepsilon)\}$.

Pick now $\varepsilon_0\in[\tau,1)$ and $u_0\in[\varepsilon_0,1]$, $v_0>0$ such that $\sup_{\tau\leqq\varepsilon<1}\{K_N\varepsilon\tau-\gamma_Y(\varepsilon)\}\leqq K_N\varepsilon_0\tau-\gamma_Y(\varepsilon_0)+K_N\tau^2$ and $F_{N,(0,\infty)}(\varepsilon_0)>(\varepsilon_0^2 N(u_0v_0)/u_0^2N(v_0))-K_N\tau^2$.

Then

$$\sigma_X(K_N\tau)\leqq K_N\left(\varepsilon_0\tau-\frac{\varepsilon_0^2 N(u_0v_0)}{u_0^2 N(v_0)}+3\tau^2\right)$$

and by Lemma 4.1 we get

$$\sigma_X(K_N\tau)\leqq K_N(\varepsilon_0\tau-(\varepsilon_0\tau-G_{M,(0,\infty)}(\tau))+3\tau^2)\leqq 4K_N\,G_{M,(0,\infty)}(\tau).$$

But from the inequality $\sigma_X(2\tau)\leqq 4\,\sigma_X(\tau)$ it follows easily that $\sigma_X(K_N\tau)\geqq(K_N^2/4)\,\sigma_X(\tau)$, so that finally we have the desired estimate

$$\sigma_X(\tau)\leqq(16/K_N^2)\,G_{M,(0,\infty)}(\tau).$$

Since the proof of (ii) and (iii) by using the inequalities (13) and (14) from Remark 4.2 above is similar, we shall not present it here.

Remark 4.8. Using the same method as in [2] one can show with Proposition 3.5 that the estimates for γ_{L_M} from Theorem 4.7 are exact in the class of all Banach spaces isomorphic to L_M.

Theorem 4.9. *The Orlicz space* $L_M(S, \Sigma, \mu)$ *with* (S, Σ, μ) *of type* (A) ((B), (C)) *is of cotype* p, $2 \leqq p < \infty$, *iff the function* $M(t^{1/p})$ *is quasiconcave, i. e.* $M((\Sigma_1^n x_i/n)^{1/p}) \geqq c\, \Sigma_1^n M(x_i^{1/p})/n$ *in* $(0, \infty)$ $([0, 1], [1, \infty))$.

Proof. We shall present a proof only for $X = L_M(S, \Sigma, \mu)$, (S, Σ, μ) of type (A). According to Theorem 4.7 X is of cotype p, iff $F_{M,(0,\infty)}(\varepsilon) \sim \varepsilon^p$, i. e.

$$M(uv) \geqq cu^p M(v), \quad u \in [0, 1], \quad v \in (0, \infty). \tag{19}$$

What we have to prove is that (19) holds iff $M(t^{1/p})$ is quasiconcave. The only 'if' part is an easy consequence of the following fact: $M(t_1^{1/p})/t_1 \geqq c\, M(t_2^{1/p})/t_2$, $t_1 \leqq t_2$, so that $M(t^{1/p})$ is equivalent to a concave function. The 'if' part can be shown with a simple analytic argument.

5. Characterization of Normed Lattices Isomorphic to Inner Product Spaces. An immediate consequence of Theorem 3.13 is the following

Theorem 5.1. *Let* X *be normed lattice. Then the following conditions are equivalent*:

(i) $\gamma_X(\varepsilon) \geqq c_1 \varepsilon^2, \quad \sigma_X(\tau) \leqq c_2 \tau^2$;

(ii) $\gamma_X(\varepsilon) \geqq c_1 \varepsilon^2, \quad \gamma_{X^*}(\varepsilon) \geqq c_3 \varepsilon^2$;

(iii) *both* X *and* X^* *are of cotype* 2;

(iv) X *is isomorphic to an inner product space.*

Proof. For the implication (i) $\Rightarrow$ (ii) we need the equality

$$\sigma_{X^{**}}(\tau) = \sigma_X(\tau). \tag{20}$$

Let $\{x_\alpha\}$, $\{y_\alpha\}$ be directed sets with $\|x_\alpha\| = \|y_\alpha\| = 1$, $Qx_\alpha \xrightarrow{w^*} x^{**}$, $Qy_\alpha \xrightarrow{w^*} y^{**}$, where Q is the canonical embedding of X into X^{**}. Then we claim that

$$\liminf_\alpha \|\sqrt{x_\alpha^2 + \tau^2 y_\alpha^2}\| \geqq \|\sqrt{x^{**2} + \tau^2 y^{**2}}\|. \tag{21}$$

Indeed, let $z_\alpha = \sqrt{x_\alpha^2 + \tau^2 y_\alpha^2}$. From $\|z_\alpha\| \leqq 1 + \tau$ it follows that we can assume with no loss of generality that $Qz_\alpha \xrightarrow{w^*} z^{**}$. For every reals a, b with $a^2 + b^2 = 1$ $ax_\alpha + b\tau y_\alpha \leqq z_\alpha$.

As the positive cone in X^{**} is w^*-closed, from the last inequality we obtain $ax^{**} + b\tau y^{**} \leqq z^{**}$. Therefore

$$\sqrt{x^{**2} + \tau^2 y^{**2}} \leqq z^{**}. \tag{22}$$

On the other hand,

$$\|z^{**}\| \leqq \liminf_\alpha \|z^\alpha\| \tag{23}$$

and from (22) and (23) it follows (21), which implies (20).

From Lemma 3.9 we have $\sigma_{X^{**}}=A_{\gamma_{X^*}}$, where A is the operator from Lemma 3.10. Using (20) and Lemma 3.10, we obtain $A\sigma_X=A^2\gamma_{X^*}\sim\gamma_{X^*}$, which completes the proof of the implication (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) follows immediately from Theorem 3.13.

(iii) $\Rightarrow$ (iv). If X and X^* are of type 2, then (see e. g. [10, p. 96]) both X, X^* are 2-concave*. X^* 2-concave implies X 2-convex* (see e. g. [10, p. 49]). But every normed lattice, which is 2-convex and 2-concave, is isomorphic to inner product space (see e. g. [10, p. 22]).

The implication (iv) $\Rightarrow$ (i) is trivial.

Remark 5.2. The condition (i) does not imply that X is isometrically isomorphic to inner product space. If $X=l_M$, where $M(x)=x^2+\sin^2 x$, then $\gamma_X(\varepsilon)\geqq c_1\varepsilon^2$, $\sigma_X(\tau)\leqq c_2\tau^2$ and X is isomorphic, but not isometrically isomorphic to l_2.

Theorem 5.3. *The normed lattice X is isometrically isomorphic to inner product space iff* $\lim_{\tau\to\infty}\gamma_X(\tau)/\sigma_X(\tau)=1$.

Proof. We start with the 'if' part. Let us denote with Y the completion of X. Then Y is a Banach lattice and $\gamma_X(\tau)=\gamma_Y(\tau)$, $\sigma_X(\tau)=\sigma_Y(\tau)$. So we have $\lim_{\tau\to\infty}\gamma_Y(\tau)/\sigma_Y(\tau)=1$. Obviously $\sqrt{1+\tau^2}-1\leqq\sigma_Y(\tau)\leqq(\sigma_Y(\tau)/(\gamma_Y(\tau))(\sqrt{1+\tau^2}-1)$, so that

$$\lim_{\tau\to 0}\tau^{-2}\sigma_Y(\tau)=1/2,$$

$$\lim_{\tau\to 0}\tau^{-2}\gamma_Y(\tau)=\lim_{\tau\to 0}\tau^{-2}(\sigma_Y(\tau))^{-1}\gamma_Y(\tau)\,\sigma_Y(\tau)=1/2.$$

Let now $x,y\in S_Y$. From $\gamma_Y(\tau)\leqq\|\sqrt{x^2+\tau^2y^2}\|-1\leqq\sigma_Y(\tau)$ it follows that

$$\lim_{\tau\to 0}\tau^{-2}(\|\sqrt{x^2+\tau^2y^2}\|-1)=1/2. \tag{24}$$

According to Kakutani's theorem (see e. g. [10, p. 15]) to prove that Y is an inner product space, it is enough to check that $\|y+z\|^2=\|y\|^2+\|z\|^2$, whenever $y\wedge z=0$, $y,z\in Y$.

Suppose $y,z\in S_Y$, $y\wedge z=0$ and let $f(t)=\|ty+z\|$ for $t>0$. Put $x=(ty+z)/f(t)$. From $y\wedge z=0$ it follows that $\sqrt{x^2+\tau^2y^2}=\sqrt{(t/f(t))^2+\tau^2}|y|+|z|/f(t)$. Therefore $\|\sqrt{x^2+\tau^2y^2}\|=f(\sqrt{t^2+f^2(t)\tau^2})/f(t)$ and from (24) we obtain

$$\lim_{\tau\to 0}\frac{f(\sqrt{t^2+f^2(t)\tau^2})-f(t)}{\tau^2f(t)}=\frac{1}{2}. \tag{25}$$

The function f is absolutely continuous, being convex. So from (25) we get $f'(t)f(t)/t=1$ and clearly $f(t)=\sqrt{1+t^2}$, i. e. $\|ty+z\|^2=1+t^2$, which completes the proof.

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* The lattice X is called p-concave (q-convex), if there exists a positive constant $M<\infty$ such that for every finite set $\{x_i\}_{i=1}^n$ of vectors in X the inequality holds

$$\Big(\sum_{i=1}^n\|x_i\|^p\Big)^{1/p}\leqq M\Big\|\Big(\sum_{i=1}^n|x_i|^p\Big)^{1/p}\Big\|\ \Big(M^{-1}\Big\|\Big(\sum_{i=1}^n|x_i|^q\Big)^{1/q}\Big\|\leqq\Big(\sum_{i=1}^n\|x_i\|^p\Big)^{1/p}\Big).$$

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