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ON THE τ-MODULUS OF SMOOTHNESS IN SOME FUNCTIONAL SPACES

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Summary. There are investigated properties of the τ-modulus $\tau_k(f;\delta)=\|\omega_k(f,\cdot;\delta)\|$, where $\omega_k(f,x;\delta)=\sup\{|\Delta_h^k f(t)|: t, t+kh\in[x-k\delta/2, x+k\delta/2]\}$, in functional spaces X with a monotone norm $\|\ \|$. Special consideration is given to the case of X — a generalized Orlicz space. A theorem of Popov [2] concerning Bernstein-type inequality for one-sided trigonometric approximation is extended to the case of bounded functions, belonging to an Orlicz space L^{φ}.

Let X be a normed space of $(b-a)$-periodic, real-valued functions with a norm $\|\ \|$ such that, if $f_1, f_2\in X$ and $|f_1(t)|\leqq|f_2(t)|$ a. e., then $\|f_1\|\leqq\|f_2\|$. The norm $\|\ \|$ will be called translation-invariant, if $f\in X$, u real imply $\|f(\cdot+u)\|=\|f\|$. For example, any generalized Orlicz space generated by a function $\varphi(t,u)$ satisfies the above monotonity condition for the norm, but only an Orlicz space generated by a function $\varphi(u)$ has always translation-invariant norm.

The following modified moduli τ_k of smoothness of order k may be defined in X: $\tau_k(f,\delta)=\|\omega_k(f,\cdot;\delta)\|$, $\delta>0$, where

$$\omega_k(f,x;\delta)=\sup\{|\Delta_h^k f(t)|: t,\ t+kh\in[x-k\delta/2,\ x+k\delta/2]\};$$

$$\Delta_h^k f(t)=\sum_{m=0}^{k}(-1)^{k+m}\binom{k}{m}f(t+mh);\quad k=1,2,3,\ldots$$

Moreover, let $\omega_k(f;\delta)=\sup\{\|\Delta_h^k f\|: 0<h\leqq\delta\}$, (see e. g. [1, 2, 3]).

1. Properties. *Let $f, g\in X$. Then*

(a) *if* $0<\delta\leqq\delta'$, *then* $\tau_k(f,\delta)\leqq\tau_k(f,\delta')$;

(b) $\tau_k(f+g;\delta)\leqq\tau_k(f;\delta)+\tau_k(g;\delta)$;

(c) $\omega_k(f;\delta/2)\leqq\tau_k(f;\delta)$.

Moreover, if the norm $\|\ \|$ is translation-invariant, then

(d) $\omega_k(f;\delta)\leqq\tau_k(f;\delta)$;

(e) $\tau_1(f;\delta_1+\delta_2)\leqq\tau_1(f;\delta_1)+\tau_1(f;\delta_2)$;

(f) $\tau_1(f;n\delta)\leqq n\tau_1(f;\delta)$, $n=1,2,\ldots$;

(g) $\tau_1(f;\lambda\delta)\leqq(\lambda+1)\tau_1(f;\delta)$, $\lambda>0$.

P r o o f. (a) follows from the obvious inequality $\omega_k(f, x; \delta) \leqq \omega_k(f, x; \delta')$ for $0<\delta \leqq \delta'$, and (b) from $\omega_k(f+g, x; \delta) \leqq \omega_k(f, x; \delta)+\omega_k(g, x; \delta)$. By the definition we have

$$|\Delta_h^k f(t)| \leqq \omega_k(f, x; \delta), \quad \text{if} \quad |t-x| \leqq k\delta/2 \quad \text{and} \quad |t+kh-x| \leqq k\delta/2.$$

If $0<h \leqq \delta/2$, then the point $t=x$ satisfies both the above inequalities and so $|\Delta_h^k f(x)| \leqq \omega_k(f, x; \delta)$. By monotonity of the norm $\| \ \|$ this implies $\| \Delta_h^k f \| \leqq \| \omega_k(f, \cdot; \delta) \| = \tau_k(f, \delta)$ for $0<h \leqq \delta/2$. Hence $\omega_k(f; \delta/2) = \sup\{\| \Delta_h^k f \| : 0<h \leqq \delta/2\} \leqq \tau_k(f; \delta)$, which proves (c).

Now, suppose $\| \ \|$ to be translation-invariant. The inequality $|\Delta_h^k f(t)| \leqq \omega_k(f, x; \delta)$ holds evidently for $t=x-k\delta/2$, $0<h \leqq \delta$, i. e. $|\Delta_h^k f(x-k\delta/2)| \leqq \omega_k(f, x; \delta)$ for $0<h \leqq \delta$.
Hence
$\| \Delta_h^k f \| = \| \Delta_h^k f(\cdot - k\delta/2) \| \leqq \| \omega_k(f, \cdot; \delta) \| = \tau_k(f; \delta)$ for $0<h \leqq \delta$, whence $\omega_k(f; \delta) = \sup\{\| \Delta_h^k f \| : 0<h \leqq \delta\} \leqq \tau_k(f; \delta)$. This proves (d).

In order to show (e), let us take two arbitrary points $t', t'' \in [x-(\delta_1+\delta_2)/2, x+(\delta_1+\delta_2)/2]$, $t'<t''$, with $\delta_1, \delta_2>0$. It is easily seen that $|f(t')-f(t'')| \leqq \omega_1(f, x+(\delta_1/2); \delta_2)$ if $t'>x-(\delta_1-\delta_2)/2$ and $|f(t')-f(t'')| \leqq \omega_1(f, x-(\delta_2/2); \delta_1)$ if $t''<x-(\delta_1-\delta_2)/2$. If $t' \leqq x-(\delta_1-\delta_2)/2 \leqq t''$, then

$$|f(t')-f(t'')| \leqq |f(t')-f(x+\frac{\delta_1-\delta_2}{2})| + |f(x+\frac{\delta_1-\delta_2}{2})-f(t'')|$$

$$\leqq \omega_1(f, x-\frac{\delta_2}{2}; \delta_1) + \omega_1(f, x+\frac{\delta_1}{2}; \delta_2).$$

Hence $\omega_1(f, x; \delta_1+\delta_2) = \sup\{|f(t')-f(t'')| : |t'-x| \leqq (\delta_1+\delta_2)/2, |t''-x| \leqq (\delta_1+\delta_2)/2\} \leqq \omega_1(f, x-(\delta_2/2); \delta_1)+\omega_1(f, x+(\delta_1/2); \delta_2)$. Since the norm $\| \ \|$ is translation-invariant, we obtain

$$\tau_1(f; \delta_1+\delta_2) = \| \omega_1(f, \cdot; \delta_1+\delta_2) \| \leqq \| \omega_1(f, \cdot -(\delta_2/2); \delta_1) \|$$

$$+ \| \omega_1(f, \cdot +(\delta_1/2); \delta_2) \| = \tau_1(f; \delta_1)+\tau_1(f; \delta_2),$$

which proves (e). Property (f) follows from (e) with $\delta_1=\delta_2=\delta$ by induction, and (g) follows from (f) in the well-known manner.

2. L e m m a. (a) *If* t, $t+kh \in [x-k\delta/2, x+k\delta/2]$, $\delta>0$, *and* f *has an absolutely continuous derivative* $f^{(k-1)}$ *of order* $k-1$, *then* $|\Delta_h^l f(t)| \leqq \delta^{l-1} \int_{x-k\delta/2}^{x+k\delta/2} |f^{(l)}(s)|\, ds$, *for* $l=1, 2, \ldots, k$;

(b) *if* $g \geqq 0$ *is integrable in* $[a, b]$ *and* $(b-a)$*-periodic, then*

$$\int_a^b \Big(\int_{x-k\delta/2}^{x+k\delta/2} g(t)dt \Big) dx \leqq (3+k\delta/(b-a))k\delta \int_a^b g(t)dt \quad \textit{for} \quad \delta>0$$

and

$$\int_a^b \Big(\int_{x-k\delta/2}^{x+k\delta/2} g(t)dt \Big) dx = k\delta \int_a^b g(t)dt \quad \textit{for} \quad 0<\delta \leqq (b-a)/k.$$

Proof. The inequality (a) with $l=1$ follows from the fact that $|\Delta_h^1 f(t)|\leq\int_{x-k\delta/2}^{x+k\delta/2}|f'(s)|\,ds$. Supposing (a) to be true for $l-1<k$, we have

$$|\Delta_h^l f(t)|\leq|\int_t^{t+h}|\Delta_h^{l-1}f'(s)|\,ds\leq|h|\delta^{l-1}\int_{x-k\delta/2}^{x+k\delta/2}|f^{(l)}(s)|\,ds\leq\delta^l\int_{x-k\delta/2}^{x+k\delta/2}|f^{(l)}(s)|\,ds,$$

because $t,\ t+h\in[x-k\delta/2,\ x+k\delta/2]$ implies $|h|\leq\delta$.

In order to prove (b), let us suppose first that $\delta\leq(b-a)/k$. Then

$$\int_a^b(\int_{x-k\delta/2}^{x+k\delta/2}g(t)dt)dx=\int_{a-k\delta/2}^{a+k\delta/2}(t+k\delta/2-a)\,g(t)dt$$

$$+\int_{a+k\delta/2}^{b-k\delta/2}k\delta g(t)dt+\int_{b-k\delta/2}^{b+k\delta/2}(b-t+k\delta/2)g(t)dt=k\delta\int_a^b g(t)dt,$$

by periodicity of g.

Now, let $k^{-1}(b-a)<\delta$. Then we have

$$\int_a^b(\int_{x-k\delta/2}^{x+k\delta/2}g(t)dt)dx\leq(b-a+k\delta/2)\int_a^b g(t)dt+(b-a)\int_{b-k\delta/2}^{a+k\delta/2}g(t)dt$$

$$+(b-a+k\delta/2)\int_a^b g(t)dt\leq 3k\delta\int_a^b g(t)dt+k\delta\int_{b-k\delta/2}^{a+k\delta/2}g(t)dt.$$

The second of the above integrals runs over an interval of length $k\delta-(b-a)$. Let r be the least positive integer such that $k\delta-(b-a)\leq r(b-a)$ and let us divide the interval $[b-k\delta/2,\ a+k\delta/2]$ in r intervals of equal length, $I_1, I_2, \ldots, I_r$. Then the length of each of these intervals is $\leq b-a$. Due to the fact that g is nonnegative and $(b-a)$-periodic, we have

$$\int_{b-k\delta/2}^{a+k\delta/2}g(t)dt=\sum_{j=1}^r\int_{I_j}g(t)dt\leq\sum_{j=1}^r\int_a^b g(t)dt=r\int_a^b g(t)dt.$$

Hence

$$\int_a^b(\int_{x-k\delta/2}^{x+k\delta/2}g(t)dt)dx\leq(3+r)k\delta\int_a^b g(t)dt.$$

But, by the definiton of the number r, we have $k\delta-(b-a)>(r-1)(b-a)$, i. e. $r<k\delta/(b-a)$. Hence

$$\int_a^b(\int_{x-k\delta/2}^{x+k\delta/2}g(t)dt)dx<(3+k\delta/(b-a))k\delta\int_a^b g(t)dt$$

for an arbitrary $\delta>0$ (since for $\delta\leq(b-a)/k$ this inequality holds, too).

In the following we shall limit ourselves to the case of a generalized Orlicz space $X=L^\varphi$. Let $\varphi:R\times R_+\to R_+$ be a function satisfying the following conditions:

(a) $\varphi(t,u)$ is a measurable, $(b-a)$-periodic function of t for every $u\geq0$;

(b) for almost every t, φ is a convex φ-function of the variable u, i. e. $\varphi(t, u)>0$ for $u>0$, $\varphi(t, 0)=0$, $\varphi(t, u)$ is a convex function of u.

Let $\rho(f)=\int_a^b \varphi(t, |f(t)|)dt$. Then the space L^φ consists of all real-valued, measurable, $(b-a)$-periodic functions f such that $\rho(\lambda f)<\infty$ for some $\lambda>0$ with the norm $\|f\|=\inf\{u>0: \rho(f/u)\leq 1\}$; if $\varphi(t, u)$ is independent of the variable t, then $\| \ \|$ is translation-invariant.

The function φ will be called approximately bounded, if there are a set $A\subset R$ of measure zero, a function $F:(R\setminus A)\times(R\setminus A)\to R_+$ and constants δ_0, k_1, $k_2>0$ such that $\varphi(x, u)\leq k_1\varphi(t, k_2u)+F(t, x)$ for every $u\geq 0$, $t\in R\setminus A$, $x\in R\setminus A$, where

$$\Phi(\delta)=\int_a^b\Big(\int_{x-\delta/2}^{x+\delta/2} F(t, x)dt\Big)dx<k\delta \quad \text{for} \quad 0<\delta<\delta_0.$$

Let us remark that: 1) if φ does not depend on the first variable, then φ is always approximately bounded with $k_1=k_2=1$, $F(t, x)=0$.

2) $F(t, x)$ satisfies the above condition, if, for example, supess $F(t, x)$ $<k(b-a)^{-1}$.

3. Theorem. *If φ is approximately bounded and f possesses a. e. the derivative $f^{(k)}\in L^\varphi$, then $\tau_k(f; \delta)\leq k_3\, k(1-\Phi(\delta)/k\delta)^{-1}\delta^k\|f^{(k)}\|$ for $0<\delta$ $<\min(\delta_0, b-a)$ with $k_3=k_2\max(1, (3+k)k_1)$.*

Proof. Applying the inequality (a) from the Lemma with $l=k$, and the definition of $\omega_k(f, x; \delta)$, we obtain

$$\omega_k(f, x; \delta)\leq\delta^{k-1}\int_{x-k\delta/2}^{x+k\delta/2}|f^{(k)}(t)|\,dt.$$

Hence, by Jensen's inequality and part (b) of the Lemma with $g(t)=$ $\varphi(t,k_2k\delta^k|f^{(k)}(t)|)$, $0<\delta\leq 2\pi$, we obtain

$$\rho(\omega_k(f, \cdot\,; \delta))\leq\frac{k_1}{k\delta}\int_a^b\Big(\int_{x-k\delta/2}^{x+k\delta/2}\varphi(t, k_2k\delta^k|f^{(k)}(t)|)dt\Big)dx$$

$$+(k\delta)^{-1}\int_a^b\Big(\int_{x-k\delta/2}^{x+k\delta/2}F(t, x)dt\Big)dx\leq(3+k)k_1\rho(k_2k\delta^k f^{(k)})+(k\delta)^{-1}\Phi(\delta).$$

Now, writing $k_3=k_2\max(1, (3+k)k_1)$, we have, by convexity of φ: $(3+k)k_1\rho(k_2k\delta^k f^{(k)})\leq\rho(k_3k\delta^k f^{(k)})$. Thus $\rho(\omega_k(f, \cdot\,; \delta))\leq\rho(k_3k\delta^k f^{(k)})+\Phi(\delta)/k\delta$ for $0<\delta<b-a$.

By the assumption $f^{(k)}\in L^\varphi$ and so taking $\delta<\delta(f)=(k_3k\|f^{(k)}\|)^{-1/k}$ we have $\|k_3k\delta^k f^{(k)}\|<k_3k\delta^k\|f^{(k)}\|<1$. Hence $\rho(k_3k\delta^k f^{(k)})\leq\|k_3k\delta^k f^{(k)}\|<1$. Then $\rho(\omega_k(f/u, \cdot\,; \delta))\leq u^{-1}k_3k\delta^k\|f^{(k)}\|+\Phi(\delta)/k\delta$ for every $u>0$, $0<\delta<\delta(f)$. The right-hand side of the above inequality is ≤ 1, if and only if
(*) $u\geq k_3k\delta^k\|f^{(k)}\|(1-\Phi(\delta)/k\delta)^{-1}$, because $\Phi(\delta)<k\delta$. Thus $\rho(\omega_k(f/u, \cdot\,; \delta))\leq 1$, if $u>0$ satisfies (*).
Hence

$$\tau_k(f; \delta)=\inf\{u>0: \rho(\omega_k(f/u, \cdot\,; \delta))\leq 1\}\leq k_3k(1-\Phi(\delta)/k\delta)^{-1}\delta^k\|f^{(k)}\|$$

for $0<\delta<\min(\delta_0, b-a)$.

4. Corollary. *If* φ *is a convex* φ*-function, independent of the parameter* t, *and if* $f^{(k)}\in L^{\varphi}$, *then* $\tau_k(f;\delta)\leq(3+k)k\delta^k\|f^{(k)}\|$ *for* $0<\delta<b-a$, *and* $\tau_k(f;\delta)\leq k\delta^k\|f^{(k)}\|$ *for* $0<\delta<(b-a)/k$.

We are going now to apply the moduli τ_k to a problem of a best one-sided approximation of functions f from an Orlicz space L^{φ} by trigonometric polynomials, with φ independent of a parameter, i. e. $\varphi(t, u)=\varphi(u)$ and $\rho(f)=\int_0^{2\pi}\varphi(|f(t)|)dt$; for analogous results in L^p see [2] for $p\geq 1$ and [3] for $0<p<1$; the last case is not included in our considerations, because we suppose φ to be convex.

Following [3], we shall write for

$$T(x)=\alpha_0/2+\sum_{k=1}^{n}\alpha_k\cos kx+\beta_k\sin kx,\quad \Lambda(T)=|\alpha_0|/2+\sum_{k=1}^{n}(|\alpha_k|+|\beta_k|).$$

The class of all trigonometric polynomials P of order $\leq n$ such that $P(x)\geq f(x)$ in $[0, 2\pi]$ will be denoted by $H_n^+(f)$ and the class of all trigonometric polynomials Q of order $\leq n$ such that $f(x)\geq Q(x)$ in $[0, 2\pi]$ — by $H_n^-(f)$. Then the best one-sided approximation of function $f\in L^{\varphi}$ is defined as

$$\widetilde{E}_n(f)=\inf\{\|P-Q\|: P\in H_n^+(f),\ Q\in H_n^-(f)\}.$$

Finally, supposing f to be bounded, we shall write

$$\|f\|_{\infty}=\text{supess}\,\{|f(x)|: x\in[a, b]\}.$$

Applying a method similar to that used in [3], we prove that denoting by φ_{-1} the inverse function to φ, we have the following

5. Lemma. *For every positive integer* m *and every essentially bounded, measurable and* 2π*-periodic function* f *there exists a constant* $M>0$ *such that if* $P\in H_n^+(f)$, $Q\in H_n^-(f)$ *and* $\Lambda(P)>M$ *or* $\Lambda(Q)>M$, *then* $\|P-Q\|>\dfrac{m}{\varphi_{-1}\left(\frac{1}{2\pi}\right)}\|f\|_{\infty}$.

Proof. Let $P\in H_n^+(f)$, $Q\in H_n^-(f)$. Then $\rho(P-Q)\geq\rho(P-f)$ and denoting by $\chi_{G(\delta)}$ the characteristic function of the set $G(\delta)=\{x\in[0, 2\pi]: P(x)\geq\Lambda(P)\delta\}$, $\delta>0$, we obtain

$$\|P-Q\|\geq\|P-f\|\geq\|P\|-\|f\|\geq\|P\chi_{G(\delta)}\|-(\varphi_{-1}(1/2\pi))^{-1}\|f\|_{\infty},$$

because

$$\|f\|\leq\|f\|_{\infty}\cdot\|1\|=\|f\|_{\infty}(\varphi_{-1}(1/2\pi))^{-1}.$$

Now, there exist $\varepsilon>0$ and $\delta>0$ such that the Lebesgue measure of $G(\delta)$ satisfies the inequality $|G(\delta)|>\varepsilon$. Hence

$$\rho(u^{-1}P\chi_{G(\delta)})\geq\int_{G(\delta)}\varphi(u^{-1}\Lambda(P)\delta)dt>\varepsilon\varphi(u^{-1}\Lambda(P)\delta)>1,$$

if $\varphi(u^{-1}\Lambda(P)\delta)>1/\varepsilon$, i. e. $u<\Lambda(P)\delta/\varphi_{-1}(1/\varepsilon)$. Hence, if $u<\Lambda(P)\delta/\varphi_{-1}(1/\varepsilon)$, then $u\leq\inf\{u>0: \rho(u^{-1}P\chi_{G(\delta)})\leq 1\}=\|P\chi_{G(\delta)}\|$. Thus $\|P\chi_{G(\delta)}\|\geq\Lambda(P)\delta/\varphi_{-1}(1/\varepsilon)$.

Consequently,

$$\| P-Q \| \geq \Lambda(P)\delta/\varphi_{-1}(1/\varepsilon)-\| f \|_\infty/\varphi_{-1}(1/2\pi).$$

Let us suppose that $\Lambda(P)>M=\delta^{-1}\varphi_{-1}(1/\varepsilon)(1+m)\| f \|_\infty(\varphi_{-1}(1/2\pi))^{-1}$. Then

$$(*)\qquad \| P-Q \| >(\varphi_{-1}(1/2\pi))^{-1}(1+m)\| f \|_\infty-(\varphi_{-1}(1/2\pi))^{-1}\| f \|_\infty$$
$$=(\varphi_{-1}(1/2\pi))^{-1}m\| f \|_\infty.$$

Similarly it may be shown that $\Lambda(Q)>M$ implies also the inequality (*). This proves the Lemma.

6. Theorem. *If f is measurable, essentially bounded and 2π-periodic, then for every n there exist $P^* \in H_n^+(f)$ and $Q^* \in H_n^-(f)$ such that $\tilde{E}_n(f)=\| P^*-Q^* \|$.*

Proof. We may suppose $\| f \|_\infty \neq 0$. Taking $m>2$ in Lemma 5, we conclude that there is an $M>0$ such that if $\| P-Q \| \leq m(\varphi_{-1}(1/2\pi))^{-1}\| f \|_\infty$, then $\Lambda(P)\leq M$ and $\Lambda(Q)\leq M$. We have $\tilde{E}_n(f)\leq \| 2 \| f \|_\infty \| = 2\| f \|_\infty \| 1 \| = 2(\varphi_{-1}(1/2\pi))^{-1}\| f \|_\infty$. Let us take $P_j \in H_n^+(f)$ and $Q_j \in H_n^-(f)$ such that

$$(**)\qquad \lim_{j\to\infty} \| P_j-Q_j \| = \tilde{E}_n(f).$$

There exists an index J such that $\| P_j-Q_j \| < \tilde{E}_n(f)+(m-2)(\varphi_{-1}(1/2\pi))^{-1} \times \| f \|_\infty$ for $j>J$. Hence $\| P_j-Q_j \| < m(\varphi_{-1}(1/2\pi))^{-1}\| f \|_\infty$ for $j>J$.

Consequently, there exists an $M>0$ such that $\Lambda(P_j)\leq M$ and $\Lambda(Q_j)\leq M$ for $j>J$. Choosing M eventually larger, these inequalities may be extended to all $j=1, 2, 3, \ldots$ Thus, there exists an increasing subsequence (j_ν) of indices such that $P_{j_\nu}(x)\to P^*(x)$, $Q_{j_\nu}(x)\to Q^*(x)$ uniformly as $\nu\to\infty$, where $P^* \in H_n^+(f)$ and $Q^* \in H_n^-(f)$. This implies $\| P_{j_\nu}-P^* \| \to 0$ and $\| Q_{j_\nu}-Q^* \| \to 0$, as $\nu\to\infty$. Hence $\| P_{j_\nu}-Q_{j_\nu} \| \to \| P^*-Q^* \|$, as $\nu\to\infty$. This together with (**) shows that $\tilde{E}_n(f)=\| P^*-Q^* \|$.

The following theorem generalizes that given in [2], Theorem 2 in case of $\varphi(u)=|u|^p$, $p\geq 1$. Let us remark that the case $0<p<1$ may be found in [3], 4.1.

7. Theorem. *Let φ be a convex φ-function without parameter and let $f \in L^\varphi$ in $[0, 2\pi]$ be bounded. Then there holds the inequality*

$$\tau_k(f; 2\pi/n)\leq C_k n^{-k} \sum_{\nu=0}^{n} (\nu+1)^{k-1} \tilde{E}_\nu(f) \quad \text{for } n=1, 2, \ldots,$$

where $C_k=k(3+k)2^{4k+4}\pi^k$.

Proof. Let $P_n^* \in H_n^+(f)$ and $Q_n^* \in H_n^-(f)$ be chosen in such a manner that $\tilde{E}_n(f)=\| P_n^*-Q_n^* \|$. Arguing as in [3], p. 153, we obtain

$$\omega_k(f, x; \delta)\leq\omega_k(P_n^*, x; \delta)+\omega_k(Q_n^*, x; \delta)$$
$$+2^k[2\omega_1(P_n^*-Q_n^*, x; k\delta)+|P_n^*(x)-Q_n^*(x)|]$$

for $\delta>0$. Hence, by property 1(f),

$$\tau_k(f; \delta)\leq\tau_k(P_n^*; \delta)+\tau_k(Q_n^*; \delta)+2^{k+1}k\tau_1(P_n^*-Q_n^*; \delta)+2^k\tilde{E}_n(f).$$

Applying a Bernstein-type inequality (see [4, Vol II, Chap. X, Th. 3.16]), we have $\rho(n^{-1}(P_n^*-Q_n^*)')\leq\rho(P_n^*-Q_n^*)$, whence $\|(P_n^*-Q_n^*)'\|\leq n\|P_n^*-Q_n^*\|$. Hence, by Corollary 4,

$$\tau_1(P_n^*-Q_n^*;\delta)\leq\delta\|(P_n^*-Q_n^*)'\|\leq n\delta\tilde{E}_n(f)$$

for $0<\delta\leq 2\pi$. Consequently,

$$\tau_k(f;\delta)\leq\tau_k(P_n^*;\delta)+\tau_k(Q_n^*;\delta)+(8\pi k+1)2^k\tilde{E}_n(f)$$

for $0<\delta\leq 4\pi/n$.

Now, we are going to estimate $\tau_k(P_{2m}^*;\delta)$. Let us write $U_0(x)=P_1^*(x)-P_0^*(x)$, $U_\nu(x)=P_{2\nu}^*(x)-P_{2\nu-1}^*(x)$, $\nu\geq 1$. Then $P_{2m}^*(x)=P_0^*(x)+\Sigma_{\nu=0}^m U_\nu(x)$ for $m\geq 1$. Hence, applying property 1(b), we obtain

$$\tau_k(P_{2m};\delta)\leq\sum_{\nu=0}^{m}\tau_k(U_\nu;\delta).$$

Applying Corollary 4, we get $\tau_k(U_\nu;\delta)\leq(3+k)k\delta^k\|U_\nu^{(k)}\|$ for $0<\delta\leq 2\pi$. From the above quoted Bernstein-type inequality we conclude $\|U_\nu^{(k)}\|\leq 2^k\|U_\nu\|$, which implies $\tau_k(U_\nu;\delta)\leq(3+k)k\delta^k 2^{k\nu}\|U_\nu\|$ for $0<\delta\leq 2\pi$. Hence

$$\tau_k(P_{2m}^*;\delta)\leq\sum_{\nu=0}^{m}\tau_k(U_\nu;\delta)\leq(3+k)k\delta^k(\|P_1^*-P_0^*\|$$

$$+\sum_{\nu=1}^{m}2^{k\nu}\|P_{2\nu}^*-P_{2\nu-1}^*\|)\leq(3+k)k\delta^k\{\|P_1^*-Q_1^*\|+\|P_0^*-Q_0^*\|$$

$$+\sum_{\nu=1}^{m}2^{k\nu}(\|P_{2\nu}^*-Q_{2\nu}^*\|+\|P_{2\nu-1}^*-Q_{2\nu-1}^*\|)\}$$

$$\leq 2(3+k)k\delta^k\{\tilde{E}_0(f)+\sum_{\nu=1}^{m}2^{k\nu}\tilde{E}_{2\nu-1}(f)\}\quad\text{for}\quad 0<\delta\leq 2\pi.$$

Similarly,

$$\tau_k(Q_{2m}^*;\delta)\leq 2(3+k)k\delta^k\{\tilde{E}_0(f)+\sum_{\nu=1}^{m}2^{k\nu}\tilde{E}_{2\nu-1}(f)\}$$

for $0<\delta\leq 2\pi$. Hence

$$\tau_k(f;\delta)\leq 4(3+k)k\delta^k\{\tilde{E}_0(f)+\sum_{\nu=1}^{m}2^{k\nu}\tilde{E}_{2\nu-1}(f)\}$$

$$+(8\pi k+1)2^k\tilde{E}_{2m}(f)\quad\text{for}\quad 0<\delta\leq 2^{-m}4\pi.$$

Since

$$\sum_{\mu=2^{\nu-2}+1}^{2^{\nu-1}}\mu^{k-1}\tilde{E}_\mu(f)\geq 2^{k\nu}\tilde{E}_{2\nu-1}(f)2^{-2k}$$

we have

$$\tau_k(f;\,\delta)\leq 4(3+k)k\delta^k\{\tilde{E}_0(f)+\tilde{E}_1(f)+2^{2k}\sum_{\mu=2}^{2^{m-1}}\mu^{k-1}\tilde{E}_\mu(f)\}$$

$$+(8\pi k+1)2^k\,\tilde{E}_{2^m}(f)\leq 8(3+k)k\delta^k 2^{2k}\sum_{\nu=0}^{2^{m-1}-1}(\nu+1)^{k-1}\tilde{E}_\nu(f)$$

$$+(8\pi k+1)2^k\tilde{E}_{2^m}(f)\quad\text{for}\quad 0<\delta\leq 2^{-m}4\pi.$$

Let an arbitrary index n be given and let us choose an integer m such that $2^{m-1}\leq n<2^m$. Let us put $\delta=2^{-m+2}4\pi$. Then we have

$$\tau_k(f;\,2\pi n^{-1})\leq\tau_k(f;\,2^{-m+1}2\pi)$$

$$\leq 8(3+k)k2^{-km}4^k\pi^k 2^{2k}\sum_{\nu=0}^{2^{m-1}-1}(\nu+1)^{k-1}\tilde{E}_\nu(f)+(8\pi k+1)2^k\tilde{E}_{2^m}(f)$$

$$\leq 8(3+k)k4^{2k}\pi^k n^{-k}\sum_{\nu=0}^{n-1}(\nu+1)^{k-1}\tilde{E}_\nu(f)+(8\pi k+1)2^k\tilde{E}_n(f).$$

But $\tilde{E}_n(f)\leq kn^{-k}\sum_{\nu=0}^{n}(\nu+1)^{k-1}\tilde{E}_\nu(f)$. Hence

$$\tau_k(f;\,2\pi/n)\leq\{8(3+k)4^{2k}\pi^k+(8\pi k+1)2^k\}kn^{-k}\sum_{\nu=0}^{n}(\nu+1)^{k-1}\tilde{E}_\nu(f)$$

$$\leq k(3+k)2^{4k+4}\pi^k n^{-k}\sum_{\nu=0}^{n}(\nu+1)^{k-1}\tilde{E}_\nu(f).$$

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