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ON SOME APPROXIMATION PROBLEMS IN MODULAR SPACES

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Summary. There is extended a theorem on modular approximation [1] to the case of approximation by nonlinear operators. The result is applied in order to obtain an approximation theorem for functions x from a generalized Orlicz space by means of integral operators of the form

$$(T_w x)(s)=\int_a^b K_w(t-s,\, x(t))\,dt$$

with respect to a filter of subsets of a set $\mathcal{W}$ of indices.

Let X be a real vector space and let ρ be a *modular* in X, i. e. $0\leqq\rho(x)\leqq\infty$, $\rho(x)=0$, iff $x=0$, $\rho(-x)=\rho(x)$, $\rho(ax+by)\leqq\rho(x)+\rho(y)$ for $x, y\in X, a, b\geqq 0, a+b=1$. If $\rho(ax+by)\leqq a\rho(x)+b\rho(y)$ for $x, y\in X, a, b\geqq 0$, $a+b=1$, then ρ is called a *convex modular*. The space $X_\rho=\{x\in X\colon \rho(ax)\to 0$, as $a\to 0\}$ is called the *modular space* generated by the modular ρ, $|x|_\rho=\inf\{u>0\colon \rho(x/u)\leqq u\}$ is an F-norm in X_ρ, and in case of convex ρ, $\|x\|_\rho=\inf\{u>0\colon \rho(x/u)\leqq 1\}$ is a norm in X_ρ. Norm convergence of (x_n) to x in X_ρ is equivalent to the condition $\rho(a(x_n-x))\to 0$ for every $a>0$. If $\rho(a(x_n-x))\to 0$ for some $a>0$, then (x_n) is called *ρ-convergent* (or *modular convergent*) to x and we write $x_n\xrightarrow{\rho}x$. A set $S\subset X_\rho$ is called *ρ-dense* in X_ρ, if for every $x\in X_\rho$ there is a sequence of $x_n\in S$ such that $x_n\xrightarrow{\rho}x$ (for this notation, see e. g. [2]).

Let $\mathfrak{W}$ be a filter of subsets of a nonempty abstract set $\mathcal{V}$. We shall write $x_v\xrightarrow{\mathfrak{W}}x$ in X_ρ, if $|x_v-x|_\rho\to 0$ or $\|x_v-x\|_\rho\to 0$ with respect to $\mathfrak{W}$, and we shall write $x_v\xrightarrow{\rho,\mathfrak{W}}x$, if $\rho(a(x_v-x))\to 0$ with respect to $\mathfrak{W}$ for some $a>0$, where $(x_v)_{v\in\mathcal{V}}$ is an indexed family of elements of X_ρ.

Let $T=(T_v)_{v\in\mathcal{V}}$ be a family of operators $T_v\colon X_\rho\to X$; we shall be interested in obtaining sufficient conditions in order that $T_v x\xrightarrow{\rho,\mathfrak{W}}x$ or $T_v x\xrightarrow{\mathfrak{W}}x$ for $x\in X_\rho$ and in applying the results in case, when X_ρ is a generalized Orlicz space. This problem was considered in case of linear operators T_v in [1] and is generalized here to the case, when T_v do not need to be linear.

Definition 1. *The family $T=(T_v)_{v\in\mathscr{V}}$, $T_v: X_\rho \to X$, will be called $\mathfrak{W}$-bounded, if there exist constants $k_1, k_2>0$ and a map $g: \mathscr{V}\to R_+=[0,\infty)$ such that $g(v)\xrightarrow{\mathfrak{W}}0$ and for every $x, y\in X_\rho$ there exists a $V_{a(x-y)}\in\mathfrak{W}$ depending on $a(x-y)$ such that*

$$\rho(a(T_vx-T_vy))\leqq k_1\rho(ak_2(x-y))+g(v)$$

for every $v\in V_{a(x-y)}$ and $a>0$.

Let us remark that, if ρ is convex, then we may always take $k_1=1$ in the above definition.

Theorem 1. *Let $T=(T_v)_{v\in\mathscr{V}}$, $T_v: X_\rho\to X_\rho$, be $\mathfrak{W}$-bounded and let S be a subset of elements $x\in X_\rho$, whose elements satisfy the condition $T_vx\xrightarrow{\mathfrak{W}}x$ in X_ρ. If S is ρ-dense in X_ρ, then $T_vx\xrightarrow{\rho,\mathfrak{W}}x$ for every $x\in X_\rho$. If S is dense in X_ρ with respect to the F-norm $|\ |_\rho$ (norm $\|\ \|_\rho$), then $T_vx\xrightarrow{\mathfrak{W}}x$ for every $x\in X_\rho$.*

Proof. We shall prove here the first part of the theorem; the second one is proved by the same arguments. Let an $\varepsilon>0$ be given arbitrarily and let $x\in X_\rho$ be given. Then there exists a $b>0$ and an $s\in S$ such that $\rho(3bk_2(x-s))<\varepsilon/6k_1$; moreover, we may suppose $\rho(3b(T_vs-s))\xrightarrow{\mathfrak{W}}0$, because $T_vs\xrightarrow{\mathfrak{W}}s$ in S. We assume $k_1, k_2\geqq 1$. Let $v\in V_{3b(x-s)}$. Then

$$\begin{aligned}\rho(b(T_vx-x))&\leqq\rho(3b(T_vx-T_vs))+\rho(3b(T_vs-s))+\rho(3b(s-x))\\&\leqq k_1\rho(3bk_2(x-s))+g(v)+\rho(3b(T_vs-s))+\rho(3b(s-x))\\&\leqq 2k_1\rho(3bk_2(x-s))+g(v)+\rho(3b(T_vs-s))<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon\end{aligned}$$

for $V\in\mathfrak{W}$, where V is chosen so small that $V\subset V_{3b(x-s)}$, $g(v)<\varepsilon/3$ and $\rho(3b(T_vs-s))<\varepsilon/3$ for $v\in V$. This proves the theorem.

Now, we shall apply the above theorem to the case of generalized Orlicz spaces. Let X be the space of all Lebesgue measurable, finite almost everywhere, extended real-valued functions in an interval $[a,b)$, $-\infty<a<b<\infty$, with equality almost everywhere. Let φ be a *φ-function with parameter*, i. e. $\varphi: [a,b)\times R\to R_+$, where φ is an even, continuous function of u, equal to zero iff $u=0$ for every $t\in[a,b)$, measurable with respect to t for every real u. If, moreover, $\varphi(t,u)$ is convex with respect to u for every $t\in[a,b)$, then it is called a *convex φ-function with parameter*. In the following we shall extend φ as a function of t to the whole R periodically with period $b-a$, writing $\varphi(t+(b-a),u)=\varphi(t,u)$ for $u\in R$, $t\in R$. Defining

$$\rho(x)=\int_a^b\varphi(t,x(t))dt\quad\text{for}\quad x\in X,$$

we obtain a modular in X and the respective modular space X_ρ is called a *generalized Orlicz space* and denoted by L^φ. If $\varphi(t,u)$ is independent of the variable t, L^φ is called an *Orlicz space*. In the following we shall extend also the functions $x \in X$ to the whole R periodically with period $b-a$, taking $x(t+(b-a))=x(t)$ for $t \in R$.

Following [1], we formulate

Definition 2. *The* φ*-function* φ *with parameter is called* τ*-bounded, if there exist constants* $k_1, k_2>0$ *and a function* $f: R \times R \to R_+$ *measurable and* $(b-a)$*-periodic with respect to the first variable such that*

$$\varphi(t-v, u) \leq k_1 \varphi(t, k_2 u) + f(t, v) \text{ for } u, v, t \in R,$$

where the function $h(v)=\int_a^b f(t, v)dt$ *is bounded in* R *and such that* $h(v) \to 0$ *as* $v \to 0$ *and as* $v \to b-a$. *We shall write* $H=\sup\{h(v): v \in R\}$.

Let us remark that if φ is a convex φ-function with a parameter, we may take $k_1=1$. Moreover, if φ does not depend on t, then it is always τ-bounded.

Definition 3. *The translation operator* τ_v: $X \to X$, $v \in R$, *is defined by the formula* $\tau_v x(t)=x(t+v)$. *The family of all translation operators will be denoted by* $\tau=(\tau_v)_{v \in R}$.

Let $\mathfrak{W}$ be the filter of neighbourhoods of zero in $\mathscr{V}=R$. Following [1], we have

Theorem 2. *If* φ *is* τ*-bounded, then* $\tau=(\tau_v)_{v \in R}$ *is* $\mathfrak{W}$*-bounded and the set* S *of simple functions in* $[a, b)$ *is* ρ*-dense in* X_ρ. *If, moreover,* $\int_a^b \varphi(t, u)dt<\infty$ *for every* $u>0$, *then* $|\tau_v s-s|_\rho \to 0$ *for every* $s \in S$.

Proof. $\mathfrak{W}$-boundedness of τ follows from the inequality $\rho(\tau_v x-\tau_v y) \leq k_1 \rho(k_2(x-y))+h(v)$ for all $v \in R$. If $x \in L^\varphi$, $x \geq 0$, we take $x_n \in S$ such that $0 \leq x_n(t) \uparrow x(t)$ and since $\int_a^b \varphi(t, cx(t))dt<\infty$ for sufficiently small $c>0$, the Lebesgue dominated convergence theorem yields $\rho(c(x_n-x)) \to 0$ as $n \to \infty$. If $x \in L^\varphi$ is arbitrary, we split x into positive and a negative part. Finally, supposing $\int_a^b \varphi(t, u)dt<\infty$ for all $u>0$ and taking $d>0$, $x=\sum_{i=1}^m c_i \chi_{A_i}$ with measurable, pairwise disjoint $A_i \subset [a, b)$, $A_i^v=(A_i-v) \dot{-} A_i$, where χ_{A_i} is the characteristic function of the set A_i, we have $|A_i^v| \to 0$ as $v \to 0$ for every i and hence

$$\rho(d(\tau_v x-x)) \leq \sum_{i=1}^{m} \int_{A_i^v} \varphi(t, dc)dt \to 0 \text{ as } v \to 0.$$

This proves $|\tau_v x-x|_\rho \to 0$.

Theorem 3. *If* φ *is* τ*-bounded and* $\int_a^b \varphi(t, u)dt<\infty$ *for every* $u>0$, *then* $\tau_v x \xrightarrow{\rho} x$ *as* $v \to 0$ *for every* $x \in L^\varphi$.

Proof follows from Theorems 1 and 2.

Now, let $\mathscr{W}$ be an abstract nonempty set and let $\mathfrak{W}$ be a filter of subsets of the set $\mathscr{W}$.

Definition 4. *A family* $K=(K_w)_{w \in \mathscr{W}}$ *of functions* K_w: $[a, b) \times R \to R$ *integrable in* $[a, b)$ *with respect to the first variable for all values of the second variable,* $K_w(t, 0)=0$, *will be called a semisingular kernel, if the functions*

$$L_w(t)=\sup_{u \neq v} \frac{|K_w(t, u)-K_w(t, v)|}{|u-v|}, \quad L(w)=\int_a^b L_w(t)dt,$$

satisfy the following conditions:

1° $0 < l = \inf_{w \in \mathcal{W}} L(w) \leq \sup_{w \in \mathcal{W}} L(w) = L < \infty$;

2° *extending* $K_w(t, u)$ *to all* $t \in R$ *periodically with period* $b-a$, *we have for some* $\delta > 0$

$$\int_a^{t_0-\delta} L_w(t)dt \xrightarrow{\mathfrak{W}} 0 \text{ and } \int_{t_0+\delta}^b L_w(t)dt \xrightarrow{\mathfrak{W}} 0$$

for every $t_0 \in (a, b)$ *of the form* $n(b-a)$ $(n=0, \pm 1, \pm 2, \ldots)$,

$$\int_{a+\delta}^{b-\delta} L_w(t)dt \xrightarrow{\mathfrak{W}} 0,$$

if $a = n(b-a)$ *for some* $n = 0, \pm 1, \pm 2, \ldots$

A semiregular kernel K *will be called singular, if*

$$r(w) = \sup_{u \neq 0} \frac{1}{|u|} \left| \int_a^b K_w(t, u)dt - u \right| \xrightarrow{\mathfrak{W}} 0.$$

We shall investigate the following family $T = (T_w)_{w \in \mathcal{W}}$ of operators

$$T_w x(s) = \int_a^b K_w(t-s, x(t))dt.$$

Theorem 4. *Let* φ *be a convex,* τ*-bounded* φ*-function with parameter and let* $K = (K_w)_{w \in \mathcal{W}}$ *be a semisingular kernel. Then* $T_w \colon L^\varphi \to L^\varphi$ *for all* $w \in \mathcal{W}$ *and* $T = (T_w)_{w \in \mathcal{W}}$ *is* $\mathfrak{W}$*-bounded.*

Proof. Let $c > 0$ be arbitrary. Then

$$\rho(c(T_w x - T_w y)) \leq \int_a^b \varphi\{s, \frac{1}{L(w)} \int_a^b L_w(t) c L(w) \,|\, x(s+t) - y(s+t) \,|\, dt\} ds$$

$$\leq \frac{1}{L(w)} \int_a^b L_w(t) \{ \int_a^b \varphi(u-t, cL \,|\, x(u) - y(u) \,|) du \} dt \leq k_1 \rho(k_2 c L(x-y)) + g(w),$$

where $g(w) = (1/L(w)) \int_a^b L_w(t) \{ \int_a^b f(u, t) du \} dt \leq l^{-1} \int_a^b L_w(t) h(t) dt$ with $0 \leq h(t) \leq H < \infty$, $h(t) \to 0$ as $t \to 0$ Let us denote by t_0 a number of the form $n(b-a)$ $(n=0, \pm 1, \pm 2, \ldots)$ such that $a \leq t_0 < b$; say, $t_0 > a$. Then we have for sufficiently small $\delta > 0$

$$\int_a^{t_0-\delta} L_w(t)dt \xrightarrow{\mathfrak{W}} 0 \quad \text{and} \quad \int_{t_0+\delta}^b L_w(t)dt \xrightarrow{\mathfrak{W}} 0$$

Hence

$$\int_a^b L_w(t)h(t)dt = \int_a^{t_0-\delta} + \int_{t_0-\delta}^{t_0+\delta} + \int_{t_0+\delta}^b = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_a^{t_0-\delta} L_w(t)h(t)dt \leq H \int_a^{t_0-\delta} L_w(t)dt \xrightarrow{\mathfrak{W}} 0$$

and similarly

$$I_3 \xrightarrow{\mathfrak{W}} 0$$

for every sufficiently small $\delta>0$, separately. Moreover, taking $\delta=\delta_0<(b-a)/2$ so small that $h(t_0+u)<\varepsilon/3L$ for $|u|<\delta$, we have

$$I_2=\int_{-\delta}^{\delta} L_w(t)h(t)dt \leqq (\varepsilon/3L)\int_a^b L_w(v)dv=\varepsilon/3.$$

Taking $\delta=\delta_0$, we may choose $W\in\mathfrak{W}$, so that $I_1<\varepsilon/3$ and $I_3<\varepsilon/3$ for $w\in W$. Hence

$$\int_a^b L_w(t)h(t)dt<\varepsilon \text{ for } w\in W, \text{ i. e. } \int_a^b L_w(t)h(t)dt \xrightarrow{\mathfrak{W}} 0.$$

Thus, $g(w)\xrightarrow{\mathfrak{W}} 0$.

In particular, taking $x\in L^\varphi$, we have $\rho(k_2cLx)<\infty$ for some $c>0$. Then

$$\rho(cT_wx)=\rho(c(T_wx-T_w0))\leqq k_1\rho(ck_2Lx)+g(w),$$

where

$$g(w)=\frac{1}{L(w)}\int_a^b L_w(t)h(t)dt\leqq\frac{H}{l}\int_a^b L_w(t)dt\leqq\frac{HL}{l}\cdot$$

Hence $\rho(cT_wx)<\infty$ and we obtain $T_wx\in L^\varphi$. Thus, $T_w\colon L^\varphi\to L^\varphi$ for every $w\in\mathscr{W}$. Moreover, taking $c=1$, we get that $T=(T_w)_{w\in\mathscr{W}}$ is $\mathfrak{W}$-bounded.

Theorem 5. *Let φ be a convex, τ-bounded (with $k_1=1$, $k_2=k\geqq 1$) φ-function with a parameter such that $\int_a^b\varphi(t,u)dt<\infty$ for every $u>0$, and let $K=(K_w)_{w\in\mathscr{W}}$ be a singular kernel. Then $T_w\colon L^\varphi\to L^\varphi$ for every $w\in\mathscr{W}$ and $T_wx\xrightarrow{\rho,\mathfrak{W}}x$ for every $x\in L^\varphi$. The following estimation holds for the error of approximation:*

$$\rho(c(T_wx-x))\leqq\frac{1}{2}\omega_\tau(2cLx,\delta)+\frac{1}{4l}(2\rho(4cLkx)+H)\{\int_a^{t_0-\delta}L_w(t)dt$$

$$+\int_{t_0+\delta}^{b}L_w(t)dt\}+\frac{1}{2}\rho(2cr(w)x),$$

where $c>0$, $w\in\mathscr{W}$, $a\leqq t_0-\delta\leqq t_0+\delta\leqq b$ and

$$\omega_\tau(y,\delta)=\sup_{|v|\leqq\delta}\int_a^b\varphi(t,y(t+v)-y(t))dt \quad \textit{for} \quad y\in X_\rho.$$

Proof. From Theorem 4 follows that $T_w\colon L^\varphi\to L^\varphi$ for all $w\in\mathscr{W}$ and $T=(T_w)_{w\in\mathscr{W}}$ is $\mathfrak{W}$-bounded. We have for every $c>0$

$$\rho(c(T_wx-x))=\int_a^b\varphi\{s,\int_a^b[K_w(t,cx(s+t))-K_w(t,cx(s))]dt$$

$$+\int_a^b K_w[t, cx(s)]dt - cx(s)\}ds \leq \frac{1}{2}\int_a^b \varphi\{s, 2\int_a^b |K_w(t, cx(s+t)) - K_w(t, cx(s))|\, dt\}ds$$

$$+\frac{1}{2}\int_a^b \varphi\{s, 2|\int_a^b K_w(t, cx(s))dt - cx(s)|\}\, ds = \frac{1}{2}J_1 + \frac{1}{2}J_2.$$

But

$$J_1 \leq \int_a^b \varphi\{s, \frac{1}{L(w)}\int_a^b L_w(t) 2cL\, |x(s+t) - x(s)|\, dt\}ds$$

$$\leq \frac{1}{L(w)}\int_a^b L_w(t)\{\int_a^b \varphi[s, 2cL\, |x(s+t) - x(s)|\,]ds\}dt$$

$$= \frac{1}{L(w)}\int_a^{t_0-\delta} + \frac{1}{L(w)}\int_{t_0-\delta}^{t_0+\delta} + \frac{1}{L(w)}\int_{t_0+\delta}^{b}.$$

Now,

$$\int_a^{t_0-\delta} L_w(t)\{\int_a^b \varphi[s, 2cL\, |x(s+t) - x(s)|\,]ds\}dt$$

$$\leq \frac{1}{2}\int_a^{t_0-\delta} L_w(t)\, [\rho(4cL\tau_t x) + \rho(4cLx)]dt \leq \int_a^{t_0-\delta} L_w(t)dt\, .\, \rho(4cLkx)$$

$$+\frac{1}{2}\int_a^{t_0-\delta} L_w(t)h(t)dt \leq (\rho(4cLkx) + \frac{1}{2}H)\int_a^{t_0-\delta} L_w(t)dt$$

and similarly

$$\int_{t_0+\delta}^{b} L_w(t)\{\int_a^b \varphi[s, 2cL\, |x(s+t) - x(s)|]ds\}dt \leq (\rho(4cLkx) + \frac{1}{2}H)\int_{t_0+\delta}^{b} L_w(t)dt.$$

Moreover,

$$\int_{t_0-\delta}^{t_0+\delta} L_w(t)\{\int_a^b \varphi[s, 2cL\, |x(s+t) - x(s)|\,]ds\}dt$$

$$= \int_{-\delta}^{\delta} L_w(u)\{\int_a^b \varphi[s, 2cL\, |x(s+u) - x(s)|]ds\}du$$

$$\leq \int_{-\delta}^{\delta} L_w(u)\rho[2cL(\tau_u x - x)]du \leq L(w)\omega_\tau(2cLx, \delta).$$

Hence

$$J_1 \leq \omega_\tau(2cLx, \delta) + \frac{1}{l}(\rho(4cLkx) + \frac{1}{2}H)\{\int_a^{t_0-\delta} L_w(t)dt + \int_{t_0+\delta}^{b} L_w(t)dt\}.$$

Next,

$$J_2 = \int_a^b \varphi\{s, 2|\frac{1}{cx(s)}(\int_a^b K_w(t, cx(s))dt - cx(s))|\, c\, |x(s)|\}ds$$

$$\leq \int_a^b \varphi(s, 2r(w)cx(s))ds = \rho(2cr(w)x).$$

Collecting the results, we obtain the estimation (+). Now, by Theorem 3, $\omega_\tau(2cLx, \delta) \to 0$ as $\delta \to 0$ for sufficiently small $c>0$. Hence by (+) and the assumptions of our theorem, $\rho(c(T_w x - x)) \xrightarrow{\mathfrak{W}} 0$ for sufficiently small $c>0$.

Example. Let $k_w\colon [a, b) \to R$ be integrable in $[a, b)$ and let us extend k_w $(b-a)$-periodically to the whole R. Let $g_w\colon R \to R$ be a family of continuous functions, satisfying the Lipschitz condition with constant g_w' in R. Then, defining $K_w(t, u) = k_w(t) g_w(u)$, we have $L_w(t) = |k_w(t)|\, g_w'$, $L(w) = g_w' \int_a^b |k_w(t)|\, dt$. Therefore, supposing $0 < l \leq g_w' \int_a^b |k_w(t)|\, dt \leq L < \infty$ for some l, L and all $w \in \mathcal{W}$ and

$$g_w' \int_a^{t_0-\rho} |k_w(t)|\, dt \xrightarrow{\mathfrak{W}} 0 \quad \text{and} \quad g_w' \int_{t_0+\delta}^{b} |k_w(t)|\, dt \xrightarrow{\mathfrak{W}} 0,$$

if $t_0 > a$, $(K_w)_{w \in \mathcal{W}}$ becomes semisingular. If, moreover,

$$\int_a^b k_w(t) dt \xrightarrow{\mathfrak{W}} 1 \quad \text{and} \quad \frac{g_w(u)}{u} \xrightarrow{\mathfrak{W}} 1 \quad \text{uniformly in } R \setminus \{0\},$$

then $(K_w)_{w \in \mathcal{W}}$ is singular. In particular, taking $g_w(u) = u$, we obtain linear integral operators (see [1]).

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