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IMBEDDING THEOREMS

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Several authors, among others Nikolskii [6], Jackson [1], Konjuskov [2], Uljanov [9], Timan [8], Stečkin [7], Leindler [3, 4, 5] have proved theorems which give conditions assuring that a function $f \in A$ should belong to another space $A_1 (\subset A)$ in terms of the modulus of continuity and in terms of the best approximation by trigonometrical polynomials.

For example Uljanov [9] proved the following

Theorem A. *If* $1 \leqq p < q$, *then* $\sum_{n=1}^{\infty} n^{q/p-2} \omega_p^q(f, n^{-1}) < \infty \Rightarrow f \in L^q$, *where* $\omega_p(f; \delta)$ *is the modulus of continuity of* f.

Furthermore Leindler [3] proved for example the following

Theorem B. *If* $1 \leqq p$, *then* $\sum_{n=1}^{\infty} n^{-1} \lambda_n \omega_p^p(f; n^{-1}) < \infty \Rightarrow f \in L^p \Lambda(L)$, *where* $\{\lambda_k\}$ *is a nondecreasing sequence, having the following property:* $\lambda_{k^2} \leqq K \lambda_k$ *for any* k, *furthermore* $\Lambda(x) = \sum_{k=1}^{[x]} k^{-1} \lambda_k$.

These results are generalized by Leindler [4] in the following way:

Theorem C. *If* $f \in \varphi(L)$, *then* $\sum_{n=1}^{\infty} n^{-2} \psi(\overline{\varphi}(n\varphi(\omega_\varphi(f; n^{-1})))) < \infty \Rightarrow f \in \psi(L)$, *where* $\psi, \varphi \in \Psi: = \{0 \leqq \psi(u): \ u^{-1}\psi(u)\uparrow, \exists\, p(>1): \ u^{-p}\psi(u)\downarrow\}$ *and* $\omega_\varphi(f; \delta) = \sup\{\overline{\varphi}(\int_0^{1-h} \varphi(|f(x+h) - f(x)|)\,dx): 0 \leqq h \leqq \delta\}$; $\overline{\varphi}$ *denotes the inverse of* φ.

Theorem D. *If* $f \in \varphi(L)$, *then* $\sum_{n=1}^{\infty} n^{-1} \lambda_n \varphi(\omega_\varphi(f; n^{-1})) < \infty \Rightarrow f \in \varphi(L)\Lambda(L)$, *where* λ_n, φ *and* Λ *have the same meaning as before.*

Later Uljanov [9] and Leindler [5] proved that Theorems A and B hold, if we replace in their conditions the modulus of continuity by the best approximation $E_n(f; p)$. Namely they proved

Theorem E. *If* $1 \leqq p < q$, *then* $\sum_{n=1}^{\infty} n^{q/p-2} E_n^q(f; p) < \infty \Rightarrow f \in L^q$.

Theorem F. *If* $1 \leqq p$, *then* $\sum_{n=1}^{\infty} n^{-1} \lambda_n E_n^p(f; p) < \infty \Rightarrow f \in L^p \Lambda(L)$.

L. Leindler raised the question of the generalization of these theorems similarly to Theorems C and D.

We proved that Theorems C and D hold, replacing the modulus of continuity by the best approximation $E_n(f; \varphi)$. Namely we proved

Theorem 1. *If* $f \in \varphi(L)$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \psi(\overline{\varphi}(n\varphi(E_n(f; \varphi)))) < \infty \Rightarrow f \in \psi(L),$$

where ψ, φ *have the same meaning as before, and* $E_n(f;\ \varphi)$ *denotes the best approximation of* f *by trigonometrical polynomials of order at most* n *in the metric of* $\varphi(L)$.

Theorem 2. *If* $f \in \varphi(L)$, *then*

$$\sum_{n=1}^{\infty} n^{-1}\lambda_n\varphi(E_n(f;\ \varphi)) < \infty \Rightarrow f \in \varphi(L)\Lambda(L),$$

where $\varphi, \Lambda, E_n(f;\ \varphi)$ *have the same meaning as before.*

Remarks. I. It is trivial, that Theorems 1 and 2 generalized the results of Theorems E and F.

II. The proofs of Theorems E and F are based on the well-known inequality

$$(*) \qquad \omega_p(f;\ n^{-1}) \leqq Kn^{-1/p} \sum_{k=1}^{n} k^{-1}k^{1/p}E_k(f;\ p).$$

In the proof of our theorems we use the following inequality proved by us:

$$\omega_\varphi(f;\ n^{-1}) \leqq K\overline{\varphi}(n^{-1}\varphi(\sum_{k=1}^{n} k^{-1}\overline{\varphi}(k\varphi(E_k(f,\ \varphi))))),$$

which is the generalization of $(*)$.

REFERENCES

1. D. Jackson. Certain problems of closest approximation. *Bull. Amer. Math. Soc.*, **39** 1933, 889-906.
2. А. А. Конюшков Наилучшие приближения тригонометрическими полиномами и коэффициенты Фурье. *Матем. сб.*, **44**, 1958, 53-84.
3. L. Leindler. On embedding of classes H_p^ω. *Acta Sci. Math.*, **31**, 1970, 13-31.
4. L. Leindler. On imbedding theorem. *Acta Sci. Math.*, **34**, 1973, 231-244.
5. L. Leindler. On imbedding of classes of functions. *Analysis Math.*, **5**, 1979, 51-65
6. С. М. Никольский. Неравенства для целых функций конечной степени и их применение в теории дифференцируемых функций многих переменных. *Труды Мат. ин-та им. В. А. Стеклова, АН СССР*, **38**, 1958, 244-278.
7. С. Б. Стечкин. О порядке наилучших приближений непрерывных функций. *Изв. АН СССР, Сер. матем.*, **15**, 1951, 219-242.
8. M. F. Timan. Orthonormal systems satisfying an inequality of S. M. Nikolskii. *Analysis Math.*, **4**, 1978, 75-82.
9. П. Л. Ульянов. Вложение некоторых классов функций. *Изв. АН СССР, Сер. матем.*, **32**, 1968, 649-686.

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