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ON EXPONENTIAL POLYNOMIALS OF THE LEAST L^p-NORM

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Let P be an algebraic polynomial of positive degree r and let $A=A(P)$ denote the annihilator of the differential operator $P(d/dt)$, i. e., the set of all complex-valued solutions of the differential equation $P(d/dt)x(t)\equiv 0$.

Fix a positive number p, $p>1$, and consider for $\Delta\geqq 0$ the function

$$F(\Delta)=F_p(P,\ \Delta)=\min\{\int_0^\Delta |t^{m(0)}-x(t)|^p\,dt:\quad x\in A\}.$$

Here and in what follows we denote $m(\lambda)$ for a complex number λ, the multiplicity of λ considered as a root of P; in particular, $m(\lambda)=0$, if $P(\lambda)\neq 0$.

It may be easily checked that the function F, obviously being positive, continuous and nondecreasing, satisfies the following inequality:

$$F(\Delta_1+\Delta_2)\geqq F(\Delta_1)+F(\Delta_2)\quad (\Delta_1,\ \Delta_2\geqq 0). \tag{1}$$

This means that, whatever polynomial P be, F is an inverse semiadditive function. Under a simple additional assumption concerning P, the latter property is more advanced, which may be seen from the following statement.

Theorem. *Assume that all roots of the polynomial P are real. Then $F_p(\Delta)$ is strictly convex for $\Delta\geqq 0$.*

Before turning to the proof we give some necessary comments.

First of all, the strict convexity is a useful property of F. Namely, it is the basic fact (c. f. [1]) in the proof of the optimality of equidistant systems of nodes in quadrature formulas on periodic classes of functions, defined by the restrictions, which are imposed in L^q, $1/p+1/q=1$, on the action of the operator $P(d/dt)$.

Furthermore, the above statement is also valid and is due to Chahkiev [1], if $p=1$. In an earlier research [2, 3] the strict convexity of F was proved under the same assumptions, concerning roots, for $p=1$ and $r=1$ or $r=2$ only. Moreover, in [1] a special case of the above theorem was also proved for all $p\in[1,\ \infty)$. Namely, proved was the strict convexity of $F_p(\Delta)$ for $m(0)=0$ and roots of P, forming an arithmetic progression $\{s\Lambda\}$, $s=1, \ldots, r$, $\Lambda=$some real constant.

For details, concerning optimal quadrature formulas, we refer to [4] and [1].

P r o o f o f t h e t h e o r e m. Let R denote the set of distinct roots of P. Consider the following set of r functions:

$$\{e^{\lambda t},\ te^{\lambda t},\ \dots,\ t^{m(\lambda)-1}e^{\lambda t}\}_{\lambda\in R} \tag{2}$$

and number it so that complete blocks in $\{\ \}$ follow one after another. The main point (see also [1]) of the proof is that the set (2), being basic in A, is in the case of purely real roots the Chebyshev, and more than that, the Markov set on each segment $[a, b]$ (c. f. [5, p. 136]). Moreover, if we number those functions, as it was said above, denote them $f_1(t), \dots, f_r(t)$ and add $f_{r+1}(t)=t^{m(0)}$ to this set, we obtain the new set of $r+1$ functions which is also Markov.

For fixed $\Delta>0$ denote $y(\Delta, t)$ the extremal 'exponential polynomial', i. e. the function of the form $y(\Delta, t)=t^{m(0)}-x(t)$, $x\in A$, for which the minimum in the definition of $F(\Delta)$ is attained. For routine reasons $y(\Delta, t)$ is uniquely defined by the following orthogonality condition:

$$\int_0^\Delta x(t)\,|y(\Delta, t)|^{p-1}\operatorname{sgn} y(\Delta, t)\,dt=0 \quad \text{for all } x\in A.$$

The latter is equivalent to the following system of nonlinear algebraic equations:

$$\int_0^\Delta f_i(t)\,|y(\Delta, t)|^{p-1}\operatorname{sgn} y(\Delta, t)\,dt=0 \quad (i=1, \dots, r). \tag{3}$$

Now we represent $y(\Delta, t)$ in the form

$$y(\Delta, t)=f_{r+1}(t)-\sum_{j=1}^{r} a_j(\Delta) f_j(t) \tag{4}$$

and observe certain properties of $y(\Delta, t)$ and those of the coefficients $a_j(\Delta)$

(i) *For each $\Delta>0$ the number of real roots of $y(\Delta, t)$ is maximal and equals r. All these roots are simple and situated strictly inside of* $(0, \Delta)$.

This is due to the above-mentioned Markov property. Furthermore, for a real vector $a=(a_1, \dots, a_r)$ let

$$x(a, t)=f_{r+1}(t)-\sum_{j=1}^{r} a_j f_j(t), \quad \sigma(a, t)=|x(a, t)|^{p-2},$$

$$G_i(a, \Delta)=\int_0^\Delta f_i(t)\sigma(a, t)x(a, t)dt \quad (i=1, \dots, r). \tag{5}$$

Then, due to (i), there is an open neighbourhood $\Omega=\Omega(\Delta)$ of the point $a(\Delta)$ in the (Euclidean) space of coefficients a, in which all $G_i(a, \Delta)$ are continuously differentiable with respect to a. Moreover, introduce for $a\in\Omega$ the scalar product of (bounded) functions φ, ψ by the relation

$$\langle\varphi, \psi\rangle=\langle\varphi, \psi\rangle_a=\int_0^\Delta \sigma(a, t)\varphi(t)\psi(t)dt. \tag{6}$$

We have $\partial G_i/\partial a_j=-(p-1)\langle f_i, f_j\rangle_a$ $(i, j=1, \dots, r)$. Thus the Jacobian of the system $G_i(a, \Delta)=0$, which is equivalent to (3), is a nonzero multiple of the Gramm matrice of the set $f_1, \ldots, f_r$ with respect to the scalar product (6). In view of (i) this Jacobian is nondegenerate and continuous in a certain open neighbourhood of each point $(a(\Delta), \Delta)$. By the theorem on implicit functions this implies (see (3)), that

(ii) *the coefficients $a_j(\Delta)$ of the extremal polynomial $y(\Delta, t)$ are continuously differentiable functions of Δ.*

By (3) these coefficients satisfy the following system of differential equations:

$$(7) \qquad \sum_{j=1}^{r} a_j'(\Delta)\langle f_i, f_j\rangle_{a(\Delta)} = c_1(\Delta) f_i(\Delta) \quad (i=1, \ldots, r)$$

with $c_1(\Delta)=|y(\Delta, \Delta)|^{p-1} \operatorname{sgn} y(\Delta, \Delta)/(p-1)$. Moreover, we easily get from (3) and (ii) that

$$(8) \qquad F'(\Delta)=|y(\Delta, \Delta)|^p=|y(\Delta, 0)|^p.$$

Since $y(\Delta, 0)\neq 0$ for $\Delta>0$ (see (i)), it follows from the above relation, (ii), (2) and (4) that

(iii) *$F''(\Delta)$ is continuous for $\Delta>0$ and*

$$(9) \qquad F''(\Delta)=c_2(\Delta)\, y'(\Delta, 0)=-c_2(\Delta) \sum_{j\in S} a_j'(\Delta)$$

with (c. f. (2)) $c_2(\Delta)=p\,|y(\Delta, 0)|^{p-1} \operatorname{sgn} y(\Delta, 0)$; $S=\{j: f_j(0)=1\}$.

Taking into account the inequality (1) and the property (iii), we see from (9) that to prove the theorem it suffices to establish the following algebraic fact:

If

$$(10) \qquad b=\sum_{j\in S} b_j,$$

where the numbers $b_j=a_j'(\Delta)$ satisfy the system (7), then $b\neq 0$.

Assume, contrary to this, that there is a positive Δ such that b, defined by (7) and (10), equals 0. Consider (7), (10) as a joint system of linear equations with respect to the unknown $b_1, \ldots, b_r, b$. That system clearly is nondegenerate.

The assumption $b=0$ leads by the Kramer rule and by $c_1(\Delta)\neq 0$ (see (i)) to existence of a nontrivial vector $(\varkappa_1, \ldots, \varkappa_r, \varkappa)$ such that the following relations hold:

$$(11) \qquad \sum_{i=1}^{r} \varkappa_i\langle f_i, f_j\rangle_{a(\Delta)} = \begin{cases} \varkappa & (j\in S), \\ 0 & (j\notin S), \end{cases}$$

$$(12) \qquad \sum_{i=1}^{r} \varkappa_i f_i(\Delta)=0.$$

Let

$$T(t)=\sum_{i=1}^{r} \varkappa_i f_i(t), \quad \Phi(\lambda) = \int_0^{\Delta} T(t)\sigma(a(\Delta), t)e^{\lambda t}dt.$$

Then (see (2)) (11), (12) can be rewritten as

$$\Phi(\lambda)=\varkappa \quad (\lambda \in R); \tag{13}$$

$$\Phi'(\lambda)=\cdots=\Phi^{(m(\lambda)-1)}(\lambda)=0 \quad (\lambda \in R,\ m(\lambda)\geqq 2); \tag{14}$$

$$T(\Delta)=0. \tag{15}$$

Relations (13) and (14) imply that

(iv) *the total number of real roots of the function*

$$\Phi'(\lambda)=\int_0^\Delta t\sigma(a(\Delta),\ t)T(t)e^{\lambda t}\,dt, \tag{16}$$

counted with their multiplicities, is not less than $r-1$.

In fact, according to (13), $\Phi(\lambda)$ takes on the constant value $\varkappa$ at each of the roots of P. Thus, by Rolle theorem, there is at least one root of $\Phi'(\lambda)$ strictly between each pair of distinct neighbouring roots of P. This and (14) prove (iv).

Since the weight-function $t\sigma(a(\Delta),\ t)$ in (16) is positive on $(0, \Delta)$, it follows from (iv) that $T(t)$ changes its sign not less than $r-1$ times on the open interval $(0, \Delta)$. By (15) there is one more root of $T(t)$ at the endpoint $t=\Delta$. Thus, the total number of real roots of $T(t)$ is not less than r. The latter means that $T(t)\equiv 0$, since $T(t)$ is a polynomial in $f_1, \ldots, f_r$, which is Markov set. It follows (see also (13)) that $\varkappa_1=\cdots=\varkappa_r=\varkappa=0$, which contradicts to our assumption and thus completes the proof.

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